













HIGHER SECONDARY  
PLANE, SOLID AND  
CO-ORDINATE GEOMETRY  
(INCLUDING MENSURATION)

= H. S. ELECTIVE MATHEMATICS, PAPER II =

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## PREFACE

This book has been written in accordance with the syllabus of Higher Secondary Course of the Board of Secondary Education, West Bengal.

Regarding the subject-matter we have tried to make the exposition clear and concise, without going into unnecessary details. A good number of typical examples have been worked out by way of illustrations and examples for exercise have been selected very carefully.

Important formulæ have been given for ready reference. A few questions of the recent years have been given at the end to give the students an idea of the standard of examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labour amply rewarded if the students find the book useful to them.

Corrections of misprints and suggestions for improvement will be thankfully received.

B. C. Das.

B. N. Mukherje

**by Profs. B. C. Das & B. N. Mukherjee**

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**(1) H. S. Trigonometry, (2) H. S. Algebra &  
(3) H. S. Plane, Solid & Co-ordinate  
Geometry with Mensuration.**

These books, written by authors who need no introduction to the teachers and the students of Mathematics, have removed the long-felt want for authoritative comprehensive text-books on *Higher Secondary Elective Mathematics*.

Besides claiming masterly treatment of the subjects and containing copious exercises capable of stimulating the interest and self-effort of young learners, these books eliminate the unnecessary trouble required for working out references, as in the case of other books in the field, written according to class-wise syllabi.

These books have the additional advantage that they present the different subjects in their entirety, thereby enabling the young learners to readily grasp the subjects without the risk of losing sight of any part of it, as is so often the case with books hitherto used by them.

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# Higher Secondary Elective Mathematics : Paper II

## PLANE GEOMETRY

(Syllabus for Class IX)

*To prove*

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle, together with twice the rectangle contained by one of those sides and the projection of the other side upon it. [ *Theorem 1 Pages 4-5* ]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle, diminished by twice the rectangle contained by one of these sides and the projection of the other side upon it. [ *Theorem 2 Pages 5-6* ]

If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally, and the converse. [ *Theorem 3 Page 17* ]

If two triangles are equiangular their corresponding sides are proportional, and the converse. [ *Theorems 5 and 6 Pages 23-24* ]

If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar. [ *Theorem 7 Page 25* ]

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally. [ *Theorem 8 and its converse Pages 35-36* ]

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each of the perpendicular are similar to the whole triangle and to one another. [ *Theorem 9 Page 37* ]

The ratio of the areas of similar triangles is equal to the ratio of the squares on the corresponding sides. [ *Theorem 10 Pages 38-39* ]

( Chapters I—II, Pages 3—48 )

# CONTENTS

| SUBJECTS                                | PAGES |
|---|-------|
| <i>Chapter I: Orthogonal Projection</i> |       |
| Orthogonal Projection                   | 3     |
| Theorem 1                               | 4     |
| Theorem 2                               | 5     |
| Apollonius' Theorem                     | 6     |
| Illustrative Examples                   | 7     |
| Exercise 1                              | 10    |
| <i>Chapter 2: Ratio and Proportion</i>  |       |
| Ratio and Proportion                    | 14    |
| Theorem 3                               | 17    |
| Theorem 4 (Converse of Theorem 3)       | 19    |
| Exercise IIA                            | 20    |
| Equiangular Triangles and Polygons      | 21    |
| Theorem 5                               | 23    |
| Theorem 6 (Converse of Theorem 5)       | 23    |
| Theorem 7                               | 25    |
| Illustrative Examples                   | 26    |
| Exercise IIB                            | 34    |
| Theorem 8                               | 35    |
| Converse of Theorem 8                   | 36    |
| Theorem 9                               | 37    |
| Theorem 10                              | 38    |
| Illustrative Examples                   | 40    |
| Exercise IIC                            | 46    |

# PLANE GEOMETRY

( Course for Class IX )

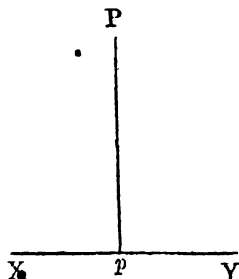
## CHAPTER I

### ORTHOGONAL PROJECTION

#### 1.1. Orthogonal Projection.

The foot of the perpendicular drawn from a given external point upon a given straight line is called the **orthogonal projection** of the given point upon the given straight line.

In the adjoining figure,  $P$  is a point outside the straight line  $XY$ , and  $Pp$  is the perpendicular drawn from the point  $P$  upon the straight line  $XY$ . Then  $p$  the foot of the perpendicular  $Pp$  on  $XY$  is the Orthogonal Projection of  $P$  on  $XY$ .



If from the extremities of a given line, straight or curved perpendiculars are drawn to a given straight line of indefinite length, the portion of the straight line intercepted between the feet of the perpendiculars is called the orthogonal projection of the given line upon the given straight line.

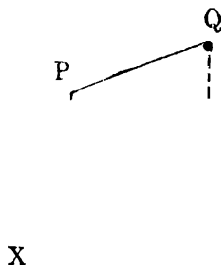


Fig. 1

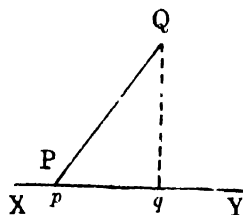


Fig. 2

In Figs. 1-3 the portion  $pq$  of the straight line  $XY$  intercepted between the feet of perpendiculars  $Pp$  and  $Qq$  drawn from the



## PLANE GEOMETRY

extremities  $P$  and  $Q$  of the straight line  $PQ$  on  $XY$  is the Orthogonal Projection of the straight line  $PQ$  on  $XY$ . In Fig. 2, the point  $P$  is on the line  $XY$ . Hence,  $p$ , the foot of the

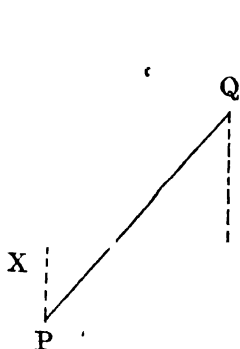


Fig. 3

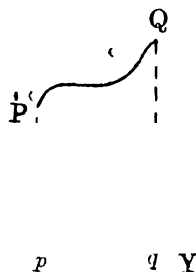
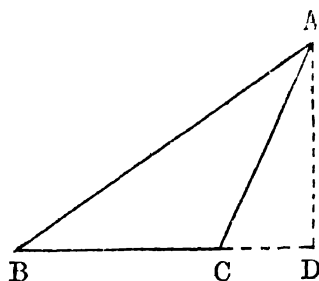


Fig. 4

perpendicular  $Pp$ , coincides with the point  $P$  of the straight line  $PQ$ . In Fig. 4, the straight line  $pq$  is the Orthogonal Projection of the curved line  $PQ$  on the straight line  $XY$ .

### Theorem 1

*In an obtuse-angled triangle the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of those sides and the projection of the other upon it.*



## ORTHOGONAL PROJECTION

Let  $ABC$  be a  $\triangle$  in which the  $\angle ACB$  is obtuse ; and from  $A$  let  $AD$  be drawn perpendicular to  $BC$  produced, so that  $CD$  is the projection of the side  $CA$  on  $BC$ .

It is reqd. to prove that  $AB^2 = BC^2 + CA^2 + 2BC.CD$ .

**Proof.** Because the  $\angle D$  is a rt. angle

$$\begin{aligned}\therefore AB^2 &= AD^2 + BD^2. && [\text{Theorem of Pythagoras}] \\ &= AD^2 + (CD + BC)^2 \\ &= AD^2 + CD^2 + BC^2 + 2BC.CD \\ &= AC^2 + BC^2 + 2BC.CD. && [\because \angle D = \text{a rt. angle}]\end{aligned}$$

### •Theorem 2

*In any triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing the acute angle diminished by twice the rectangle contained by one of those sides and the projection of the other side upon it.*

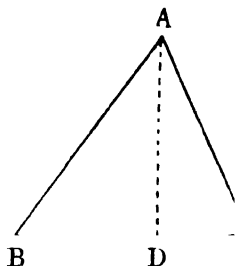


Fig. 1

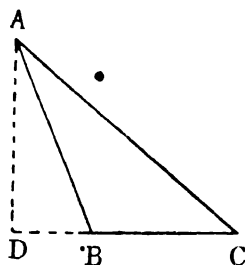


Fig. 2

Let  $ABC$  be a  $\triangle$  in which the  $\angle ACB$  is acute ; and from  $A$  let  $AD$  be drawn perp. to  $BC$  or  $BC$  produced, so that  $CD$  is the projection of  $CA$  on  $BC$ .

It is reqd. to prove that  $AB^2 = AC^2 + BC^2 - 2BC.CD$ .

**Proof.** Because the  $\angle D$  is a rt. angle,

$$\begin{aligned}\therefore AB^2 &= AD^2 + BD^2 && [\text{Theorem of Pythagoras}] \\ &= AD^2 + (BC - CD)^2 && [\text{In Fig. 1, } BD = BC - CD]\end{aligned}$$

## PLANE GEOMETRY

$$\begin{aligned}
 \text{or,} \quad &= AD^2 + (CD - BC)^2 \quad [\text{In Fig. 2, } BD = CD - BC] \\
 &= AD^2 + CD^2 + BC^2 - 2BC \cdot CD \\
 &= AC^2 + BC^2 - 2BC \cdot CD. \quad [\because \angle D \text{ is a rt. angle}]
 \end{aligned}$$

**N. B.** In the previous two theorems, it is noticed that in the  $\triangle ABC$ ,

if (i)  $AB^2 > AC^2 + BC^2$ , the  $\angle C$  will be obtuse,

and (ii)  $AB^2 < AC^2 + BC^2$ , the  $\angle C$  will be acute.

It is needless to point out that if  $AB^2 = AC^2 + BC^2$ , then according to the Converse of the Theorem of Pythagoras, the  $\angle C$  will be a rt. angle.

### An Important Deduction : Apollonius' Theorem.

*In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.*

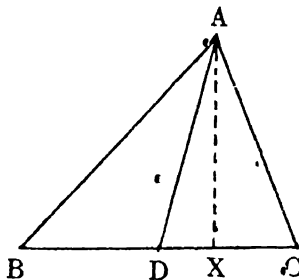


Fig. 1

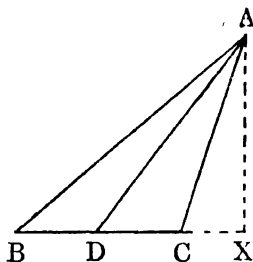


Fig. 2

In the  $\triangle ABC$  let  $AD$  be the median bisecting the sides  $BC$  at the point  $D$ .

It is reqd. to prove that  $AB^2 + AC^2 = 2(BD^2 + AD^2)$ .

**Construction.** From the pt.  $A$  draw  $AX$  perp. to  $BC$  or  $BC$  produced.

**Proof.** Suppose  $AB$  and  $AC$  are unequal. Then of the two angles  $ADB$  and  $ADC$  one is obtuse and the other is acute. Let the  $\angle ADB$  be obtuse, then the  $\angle ADC$  will be acute.

$\therefore$  from the  $\triangle ADB$ ,  $AB^2 = BD^2 + AD^2 + 2BD \cdot DX$ .

[ Theor. 1 ]

and from the  $\triangle ADC$ ,  $AC^2 = CD^2 + AD^2 - 2CD \cdot DX$ . [ Theor. 2 ]

## ORTHOGONAL PROJECTION

Adding these two results and remembering that  $BD = CD$

( $\because D$  is the middle pt. of  $BC$ ), we have

$$AB^2 + AC^2 = 2(BD^2 + AD^2).$$

N. B. If  $AB = AC$ , the proof of the theorem is easy enough.

### 12. Illustrative Examples.

Ex. 1. The  $\angle ACB$  of a  $\triangle ABC$  is  $60^\circ$ ; prove that

$$AB^2 = AC^2 + BC^2 - AC \cdot BC.$$

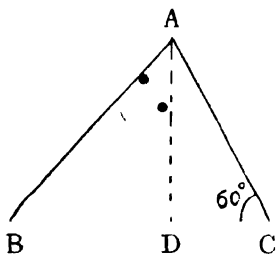


Fig. 1

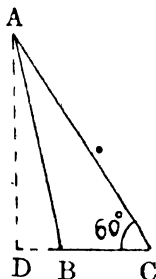


Fig. 2

Let  $ABC$  be a  $\triangle$  in which the  $\angle ACB = 60^\circ$ .

It is reqd. to prove that  $AB^2 = AC^2 + BC^2 - AC \cdot BC$ .

**Construction.** From the pt.  $A$  draw  $AD$  perp. to  $BC$  or  $BC$  produced.

**Proof.** In the rt.-angled  $\triangle ADC$ , the  $\angle ACD = 60^\circ$  (Hyp.)

$$\therefore \angle DAC = 30^\circ.$$

$\therefore$  the side  $CD$  opposite to the  $\angle DAC = \frac{1}{2}AC$ . [In a rt.-angled  $\triangle$  it can be easily proved that if one of its angles be  $30^\circ$ , the side opposite to the  $30^\circ$  angle is equal to half the hypotenuse.]

Now, in the  $\triangle ACB$ , the  $\angle ACB$  is acute.

$$\therefore AB^2 = AC^2 + BC^2 - 2BC \cdot CD. \quad [\text{Theor. 2}]$$

$$= AC^2 + BC^2 - AC \cdot BC. \quad [\because CD = \frac{1}{2}AC]$$

**Ex. 2.** Prove that a triangle whose sides are 12 cms., 14 cms., and 18 cms. is an acute-angled triangle.

Here the sides 18 cms. is the greatest side of the  $\triangle$ .

$\therefore$  the  $\angle$  opposite to this side must be the greatest  $\angle$  of the  $\triangle$ .

Now, if it can be proved that the  $\angle$  opposite to this side is acute, then

the other two angles of the  $\triangle$  being less than this angle must also be acute and the  $\triangle$  will be an acute-angled one.

From Theorem 2, we know that in any triangle the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle.

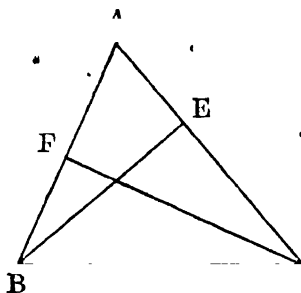
$$\text{Now } 18^2 < 12^2 + 14^2.$$

$\therefore$  the angle opposite to the 18 cms. side is an acute angle.

$\therefore$  the other two angles of the  $\triangle$  being less than this angle must also be acute.

$\therefore$  all the three angles of the  $\triangle$  being acute, it must be an acute-angled one.

**Ex. 3.** The angles  $B$  and  $C$  of a triangle  $ABC$  are acute and  $BE, CF$  are perpendiculars to  $AC, AB$  respectively; prove that  $BC^2 = AB \cdot BF + AC \cdot CE$ .



Let  $ABC$  be a  $\triangle$  in which the  $\angle B, C$  are acute and  $BE, CF$  are drawn perps. to  $AC, AB$  respectively.

It is reqd. to prove that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

**Proof.** Since the  $\angle C$  is acute,

$$AB^2 = AC^2 + BC^2 - 2AC \cdot CE.$$

$$\therefore BC^2 = AB^2 - AC^2 + 2AC \cdot CE.$$

In a similar way, since the  $\angle B$  is acute

$$BC^2 = AC^2 - AB^2 + 2AB \cdot BF.$$

Adding these two results,  $2BC^2 = 2AC \cdot CE + 2AB \cdot BF$ ,

$$\text{that is, } BC^2 = AC \cdot CE + AB \cdot BF.$$

**Ex. 4.** If  $O$  is any point within a rectangle  $ABCD$ , show that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

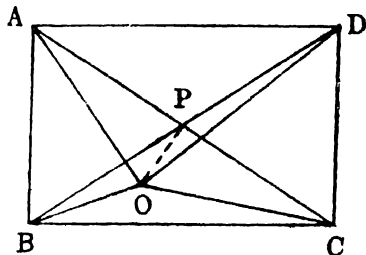
Let  $O$  be any point within the rect.  $ABCD$ . Join  $OA, OB, OC, OD$ .

It is reqd. to prove that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

**Construction.** Join  $AC, BD$  and let  $P$  be the pt. of their intersection. Join  $OP$ .

**Proof.** The diagonals  $AC, BD$  of the rect.  $ABCD$  bisect each other at  $P$ .



$\therefore$  in the  $\triangle AOC$ ,  $OA^2 + OC^2 = 2(AP^2 + OP^2)$  } [ Apollonius' Theorem ]  
 and in the  $\triangle BOD$ ,  $OB^2 + OD^2 = 2(BP^2 + OP^2)$  }

But  $AP = BP$  [  $\because AC, BD$  being the diagonal of a rect. are equal ]

$$\therefore OA^2 + OC^2 = OB^2 + OD^2.$$

**Ex. 5.** Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonal.

Let the diagonals  $AC, BD$  of the par<sup>m</sup>.  $ABCD$  intersect at  $P$ .

It is reqd. to prove that  $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ .

**Proof.** Since the diagonals of a par<sup>m</sup>. bisect each other,

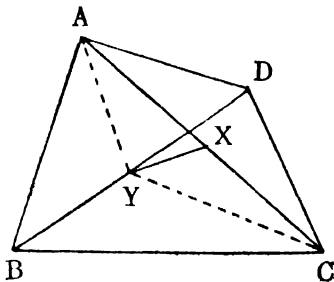
$$\therefore AP = CP \text{ and } BP = DP.$$

Now, in the  $\triangle ABC$ ,  $AB^2 + BC^2 = 2(AP^2 + BP^2)$  } [ Apollonius' Theorem ]  
 and in the  $\triangle ADC$ ,  $CD^2 + DA^2 = 2(AP^2 + DP^2)$  }

$\therefore$  by adding these two results,

$$\begin{aligned} AB^2 + BC^2 + CD^2 + DA^2 &= 4AP^2 + 2BP^2 + 2DP^2 \\ &= 4AP^2 + 4BP^2 \quad [\because BP = DP] \\ &= (2AP)^2 + (2BP)^2 \\ &= AC^2 + BD^2. \end{aligned}$$

**Ex. 6.** Prove that in any quadrilateral the sum of the squares on its sides is equal to the sum of the squares on its diagonals together with four times the square on the straight line joining the middle points of the diagonals.



Let the middle points of the diagonals  $AC, BD$  of the quadrilateral  $ABCD$  be  $X, Y$  respectively. Join  $XY$ .

It is reqd. to prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2.$$

**Construction.** Join  $AY, CY$ .

**Proof.** In the  $\triangle ABD$ ,  $AB^2 + AD^2 = 2(BY^2 + AY^2)$  } [ Apollonius' Theo. ]  
 and in the  $\triangle BCD$ ,  $BC^2 + CD^2 = 2(BY^2 + CY^2)$  }

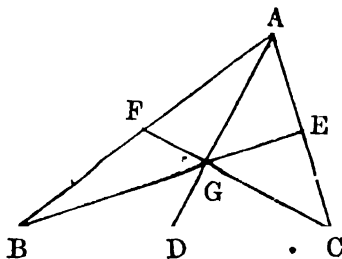
$\therefore$  by adding  $AB^2 + AD^2 + BC^2 + CD^2 = 4BY^2 + 2(AY^2 + CY^2)$ .

Again in the  $\triangle ACX$ ,  $AX^2 + CY^2 = 2(AX^2 + 2XY^2)$  [Apollonius' Theorem]

$$\begin{aligned}\therefore AB^2 + BC^2 + CD^2 + DA^2 &= 4BY^2 + 4AX^2 + 4XY^2 \\ &= (2BY)^2 + (2AX)^2 + 4XY^2 \\ &= BD^2 + AC^2 + 4XY^2.\end{aligned}$$

**Ex. 7.** If  $G$  is the centroid of a triangle  $ABC$ , prove that  $AB^2 + AC^2 + BC^2 = 3(GA^2 + GB^2 + GC^2)$ .

[The point of intersection of the three medians of a triangle is called the **Centroid** of the triangle.]



Let  $G$  be the pt. of intersection of the three medians  $AD$ ,  $BE$ ,  $CF$  of the  $\triangle ABC$ , so that  $G$  is the centroid of the  $\triangle$ .

It is reqd. to prove that

$$\begin{aligned}AB^2 + AC^2 + BC^2 \\ = 3(GA^2 + GB^2 + GC^2).\end{aligned}$$

**Proof.** By Apollonius' Theorem, we have

$$AB^2 + AC^2 = 2(BD^2 + AD^2)$$

$$BC^2 + AC^2 = 2(AF^2 + CF^2)$$

$$\text{and } AB^2 + BC^2 = 2(AF^2 + BE^2).$$

Adding these results, we get

$$2(AB^2 + BC^2 + AC^2) = 2(BD^2 + AF^2 + AE^2) + 2(AD^2 + BE^2 + CF^2)$$

$$\begin{aligned}\text{that is, } 4(AB^2 + BC^2 + AC^2) &= 4(BD^2 + AF^2 + AE^2) + 4(AD^2 + BE^2 + CF^2) \\ &= (2BD)^2 + (2AF)^2 + (2AE)^2 + 4(AD^2 + BE^2 + CF^2) \\ &= BC^2 + AB^2 + AC^2 + 4(AD^2 + BE^2 + CF^2).\end{aligned}$$

$$\therefore 3(AB^2 + BC^2 + AC^2) = 4(AD^2 + BE^2 + CF^2).$$

Since  $G$  is the pt. of trisection of the medians,  $GA = \frac{2}{3}AD$ ,  $GB = \frac{2}{3}BE$  and  $GC = \frac{2}{3}CF$ .

$$\therefore 4AD^2 = 9GA^2, 4BE^2 = 9GB^2 \text{ and } 4CF^2 = 9GC^2.$$

$$\therefore 4(AD^2 + BE^2 + CF^2) = 9(GA^2 + GB^2 + GC^2).$$

$$\text{Hence, } 3(AB^2 + BC^2 + AC^2) = 9(GA^2 + GB^2 + GC^2).$$

$$\text{i.e. } AB^2 + BC^2 + AC^2 = 3(GA^2 + GB^2 + GC^2).$$

**Ex. 8.** Prove that the locus of a point, which moves so that the sum of the squares of its distances from two fixed points is constant, is a circle.

Let  $A$  and  $B$  be two fixed pts. and  $P$  a moving point, such that  $PA^2 + PB^2 = a \text{ constant}$ .

It is reqd. to prove that the locus of  $P$  is a circle.

**Construction.** Join  $AB$  and let  $O$  be the middle pt. of  $AB$ .

**Proof.** By Apollonius' Theorem,

$$PA^2 + PB^2 = 2(AO^2 + PO^2) = \text{a constant.}$$

But  $A$  and  $B$  being fixed pts., the middle pt.  $O$  of  $AB$  is also a fixed pt.

$\therefore AO = \text{a constant.}$

Again, since  $AO^2 + PO^2$  is constant (proved),  $PO$  must also be constant, i.e. the distance of the moving pt.  $P$  from the fixed pt.  $O$  is always the same.

Hence, the locus of  $P$  is a circle with centre at  $O$ , the middle pt. of  $AB$ .

**Ex. 9.** In a triangle  $ABC$ ,  $X$  is a fixed point of trisection of the base  $BC$  adjacent to the angle  $B$ ; prove that  $2AB^2 + AC^2 = 6BX^2 + 3AX^2$ .

Let  $X$  be a pt. in the base  $BC$  of a triangle  $ABC$  such that  $BX = \frac{1}{3}BC$ .

Join  $AX$ .

It is reqd. to prove that

$$2AB^2 + AC^2 = 6BX^2 + 3AX^2.$$

**Construction.** Bisect  $XC$  at  $Y$ . Join  $AY$ .

Then  $BX = XY = YC$ .

**Proof.** Now, in the  $\triangle ABY$ ,  $AB^2 + AY^2 = 2(BX^2 + AX^2)$  [Apollonius' Theorem]

$$\therefore 2AB^2 + 2AY^2 = 4BX^2 + 4AX^2.$$

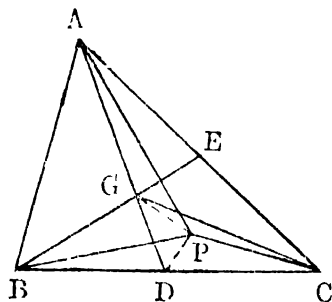
Again, in the  $\triangle AXC$ ,  $AX^2 + AC^2 = 2(XY^2 + AY^2)$  [Apollonius' Theorem]

Adding the results, we have

$$2AB^2 + 2AY^2 + AX^2 + AC^2 = 4BX^2 + 4AX^2 + 2XY^2 + 2AY^2.$$

$$\therefore 2AB^2 + AC^2 = 6BX^2 + 3AX^2, \text{ since } BX = XY.$$

**Ex. 10.** If  $G$  is the centroid of a triangle  $ABC$  and  $P$  any point, prove that  $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$ .



Let  $G$  be the pt. of intersection of the two medians  $AD, BE$  of the  $\triangle ABC$  and  $P$  any point. Then  $G$  is the centroid of the  $\triangle ABC$ . Join  $PA, PB, PC, GP$  and  $GC$ .

It is reqd. to prove that  $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$ .

**Construction.** Join  $PD$ .

**Proof.** Since  $AG = 2GD$ , with the help of the preceding Ex. 9, from the  $\triangle APD$ , we get  $2PD^2 + PA^2 = GD^2 + 3GP^2$ .



Now,  $PB^2 + PC^2 = 2BD^2 + 2PD^2$ . [ Apollonius' Theorem ]

Adding these two results, we have

$$2PD^2 + PA^2 + PB^2 + PC^2 = 6GD^2 + 3GF^2 + 2BD^2 + 2PD^2$$

$$\begin{aligned} \text{i.e.} \quad PA^2 + PB^2 + PC^2 &= 6GD^2 + 2BD^2 + 3GP^2 \\ &= 4GD^2 + 2(GD^2 + BD^2) + 3GP^2 \\ &= GA^2 + GB^2 + GC^2 + 3GP^2. \end{aligned}$$

$$\therefore GA = 2GD].$$

N.B. There will be no change in the proof, if the point  $P$  be taken outside the circle.

### Exercise I

1.  $ABC$  is an isosceles triangle in which  $AB = AC$ , and  $CD$  is drawn perpendicular to  $AB$ ; prove that  $BC^2 = 2AB \cdot BD$ .

2. The angle  $ACB$  of a triangle  $ABC$  is  $120^\circ$ ; prove that  
 $AB^2 = AC^2 + BC^2 + AC \cdot BC$ .

3. Show that a triangle whose sides are 17 cm., 13 cm. and 11 cm. is an acute-angled triangle.

4. The sides of a triangle are 13, 35 and 43. Show that it is an obtuse-angled triangle, and that the obtuse angle is  $120^\circ$ .

5. The base of a triangle is 12 cm. and the sum of the squares on the other two sides is 234 sq. cm.; find the locus of the vertex.

6. The base  $BC$  of an isosceles triangle is produced to  $D$  making  $CD = BC$ ; prove that  $AD^2 = AB^2 + 2BC^2$ .

7.  $O$  is the point of intersection of the diagonals of a square  $ABCD$  and  $P$  is any other point. Prove that

$$AP^2 + BP^2 + CP^2 + DP^2 = 4OA^2 + 4OP^2.$$

8.  $ABC$  is an acute-angled triangle and  $AD$ ,  $BE$  and  $CF$  are perpendiculars to the opposite sides  $BC$ ,  $CA$  and  $AB$ ; show that  $AB^2 + BC^2 + CA^2 = 2(AB \cdot AF + BC \cdot BD + AC \cdot CE)$ .

9. If the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals, prove that the quadrilateral is a parallelogram.

10. Prove that in any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the straight lines joining the middle points of the opposite sides.

11. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians. Prove this.

12.  $ABC$  is an isosceles triangle and  $A$  is joined to any point  $D$  in the base  $BC$  or  $BC$  produced;  $DA$  is produced to  $E$  making  $AE = AD$ ; show that the difference of the squares on  $BE$  and  $CE$  is equal to the difference of the squares on  $BD$  and  $CD$ .

13. The sum of the squares on the diagonals of a trapezium is equal to the sum of the squares on the two non-parallel sides together with twice the rectangle contained by the two parallel sides. Prove this.

14. In any triangle prove that the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertex to the base.

## CHAPTER II

### RATIO AND PROPORTION

#### 2.1. Ratio and Proportion.

The number which expresses the relation which one quantity bears to another of the same kind is called the **Ratio** of the two quantities. The ratio expresses what multiple or part one quantity is of the other.

To determine what multiple or what part is one straight of another in length, we divide the length of the first by the length of the second, their length being expressed in the same unit. In a similar manner to find how many times a quadrilateral is of a triangle in area, we divide the area of the quadrilateral by the area of the triangle, both expressed in the same unit. Hence, the ratio is an abstract number and not a concrete one.

Suppose there are two straight lines 12 cms. and 8 cms. long. To determine how many times the first of them is greater than the second, we divide 12 cms. by 8 cms. and get the absolute number  $\frac{12}{8}$  or  $\frac{3}{2}$ . Hence the ratio of the lengths of these two straight lines is  $\frac{3}{2}$  and this is usually written as 3 : 2. If the measure of two quantities of the same kind be 'a' units and 'b' units respectively, we determine their ratio by dividing 'a' units by 'b' units, which is usually written as  $a \div b$ . By omitting the ' $\div$ ' from the sign of division ' $\div$ ' we write ':' as the sign of a ratio.

With the help of a ratio we compare two quantities of the same kind. As two quantities cannot be compared unless they are of the same kind, so the ratio of two quantities cannot be ascertained unless they are of the same kind. Again, if two quantities of the same kind are expressed in different units; to compare the one with the other both of them have got to be expressed in the same unit. For example, there are two straight lines 4 ft. and 8 in. long. Their ratio will not be denoted by 4 : 8 or 1 : 2. Both of them have to be reduced either to feet or to inches and then the ratio of the two has to be determined. Hence the ratio of the lengths of these two straight lines will be denoted by 48 : 8 when expressed in inches or by  $4 \cdot \frac{2}{3}$  when expressed in feet and both of them will give the same result 6 : 1.

The two quantities 'a' and 'b' of the ratio  $a : b$  are called the terms of the ratio. The first term 'a' is called the **antecedent** and the second term 'b' is called the **consequent**.

### Commensurable and Incommensurable Quantities

If two quantities can be expressed as multiples of some common factor as unit, i.e. can be completely expressed by the ratio of two integers, the quantities are said to be **commensurable** to each other; and quantities that cannot be so expressed are said to be **incommensurable** quantities.

The two quantities 6 ft. and 10 ft. are commensurable to each other as the ratio of these two quantities can be expressed by the ratio of the two integers 6 and 10. If the length of the side of a square is 1 cm., then the length of the diagonal is  $\sqrt{2}$  cms. Hence, the sides of the square is not commensurable with the diagonal as the ratio of the two  $1 : \sqrt{2}$  cannot be expressed by two integers; or in other words, the sides of a square and its diagonal are incommensurable quantities.

### Proportion

Four quantities are said to be **proportionals** or in **proportion** when the ratio of the first to the second is equal to the ratio of the third to the fourth, or in other words when two ratios are equal, the four quantities forming the two ratios are called proportionals. Suppose, the ratio of 'a' to 'b' is equal to the ratio of 'c' to 'd', then the four quantities, a, b, c, d are proportionals or in proportion. This is usually expressed by  $a : b = c : d$  or  $a : b :: c : d$ .

Of the four quantities a, b, c, d forming the two ratios, the first 'a' and the fourth 'd' are called the **extremes** and the two middle ones 'b' and 'c' are called the **means**. The fourth quantity 'd' is called the **fourth proportional** to a, b, c.

If three quantities a, b, c be such that the ratio of the first to the second is equal to the ratio of the second to the third, i.e. if  $a : b = b : c$ , then the three quantities a, b, c are said to be in *continued proportion*, the second quantity 'b' is called the **mean proportional** between 'a' and 'c' and the third quantity 'c' is called the **third proportional** to 'a' and 'b'.

Ratio being an absolute number, all the four quantities forming the two ratios of a proportion may be of one kind or the first two quantities forming the first ratio may be of one kind and the remaining two quantities forming the second ratio of a different kind. This will be made clear by an example. Without committing any error, we can say that the ratio of Rs. 10 to Rs. 5 is equal to the ratio of 6 hrs. to 3 hrs., since both of them are equal to 2 : 1. Hence, we can rightly say that Rs. 10, Rs. 5, 6 hrs. and 3 hrs. are in proportion.

### Some Important Fundamental Theorems on Proportion.

If four quantities  $a, b, c, d$  are in proportion, several other useful proportions can be derived from this by making suitable operations on the given proportion. The method applied to arrive at a particular result is given in any book on Algebra. In Geometry we are only concerned with the final result obtained after the operation. The different results go by different names as given below without showing how they are obtained.

(i) If  $a, b, c, d$  be in proportion i.e.  $a : b = c : d$ , then

(1)  $ad = bc$ ; this operation is known as **Cross-multiplication**;

(2)  $\frac{a}{c} = \frac{b}{d}$ ; this operation is known as **Alternando**;

(3)  $\frac{b}{a} = \frac{d}{c}$ ; this operation is known as **Invertendo**;

(4)  $\frac{a+b}{b} = \frac{c+d}{d}$ ; this operation is known as **Componendo**;

(5)  $\frac{a-b}{b} = \frac{c-d}{d}$ ; this operation is known as **Dividendo**;

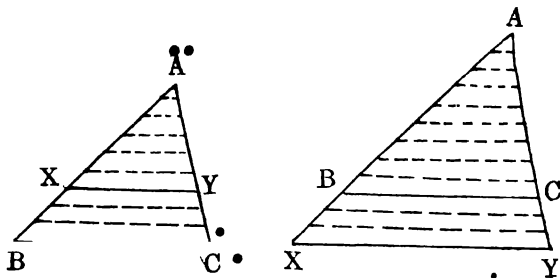
and (6)  $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ ; this operation is known as **Componendo-Dividendo**.

(ii) If  $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \dots$ , then each of these ratios

$= \frac{a+b+c+\dots}{x+y+z+\dots}$ ; this operation is known as **Addendo**.

### ✓ Theorem 3

*If a straight line is drawn parallel to one side of a triangle, it divides the other two sides or those sides produced proportionately.*



Let the straight line  $XY$  be drawn  $\parallel$  to the side  $BC$  of the  $\triangle ABC$  to cut the sides  $AB$  and  $AC$  or those sides produced at  $X$  and  $Y$  respectively.

It is required to prove that  $AX : XB = AY : YC$ .

**Proof.** Let the str. line  $XY$  be drawn  $\parallel$  to the side  $BC$  of the  $\triangle ABC$  to cut the sides  $AB$  and  $AC$ , or those sides produced, at  $X$  and  $Y$  respectively.

Let the str. lines  $AX$  and  $XB$  be commensurable to each other and let  $m$  be their common measure. Then each of the str. lines  $AX$  and  $XB$  can be divided into some equal parts, each equal to ' $m$ '. Suppose  $AX$  is divided into  $p$  equal parts, and  $XB$  into  $q$  equal parts, each equal to ' $m$ '.

$$\therefore AX = pm \text{ and } XB = qm.$$

$$\therefore AX : XB = pm : qm = p : q.$$

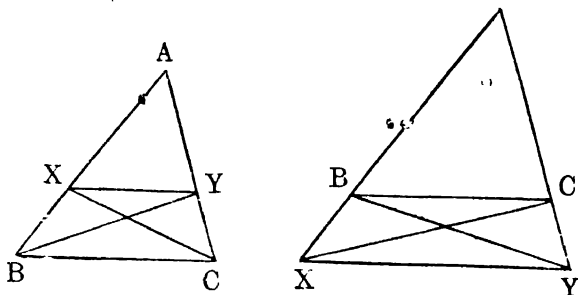
Now, through the pts. of division in  $AX$  and  $XB$  draw str. lines  $\parallel$  to the side  $BC$ . These  $\parallel$  str. lines will divide  $AY$  and  $YC$  into  $p$  and  $q$  equal parts respectively.

Let the length of each of these equal parts be  $n$ .

$$\text{Then } AY = pn \text{ and } YC = qn.$$

$$\therefore AY : YC = pn : qn = p : q.$$

$$\therefore AX : XB = AY : YC.$$

**Alternative Method.**

**Construction.** Join  $BY$ ,  $CX$ .

**Proof.** Since  $BC$  and  $XY$  are parallel,

$$\triangle BXY = \triangle CXY \text{ [ on the same base } XY \text{ ]}$$

$$\therefore \triangle AXY : \triangle BXY = \triangle AXY : \triangle CXY.$$

Now, the areas of two  $\triangle$ 's of equal altitudes are to one another as their bases.

$$\begin{aligned} \therefore \triangle AXY : \triangle BXY &= AX : XB. & [\text{the vertex being the same}] \\ \text{and } \triangle AXY : \triangle CXY &= AY : YC. & [\text{in the two cases, the } \triangle^s \\ & & \text{are of equal altitudes.}] \end{aligned}$$

**Corollary.** If the str. line  $XY$  drawn  $\parallel$  to the side  $BC$  cuts  $AB$ ,  $AC$  at  $X$ ,  $Y$  respectively, then  $AB : AX = AC : AY$ .

$$\text{Now, by Theorem 3, } \frac{AX}{XB} = \frac{AY}{YC},$$

$$\frac{XB}{AX} = \frac{YC}{AY} \text{ [ by Invertendo ]}$$

$$\therefore \text{ in Fig. 1, } \frac{XB + AX}{AX} = \frac{YC + AY}{AY} \text{ [ by Componendo ]}$$

$$\text{and in Fig. 2, } \frac{AX - XB}{AX} = \frac{AY - YC}{AY} \text{ [ by Dividendo ]}$$

$$\frac{AB}{AX} = \frac{AC}{AY}, \text{ i.e. } AB : AX = AC : AY.$$

**N. B.** In a similar way it can be proved in this case that

$$AB : BX = AC : CY.$$

**Theorem 4**

[ CONVERSE OF THEOREM 3 ]

*If a straight line cuts two sides of a triangle, or those two sides produced, proportionately, then it is parallel to the remaining side of the triangle.*

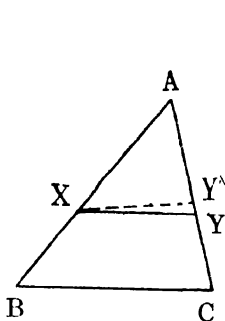


Fig. 1

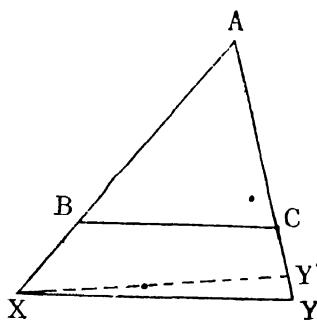


Fig. 2

In the  $\triangle ABC$  let the str. line  $XY$  be drawn to cut the sides  $AB$  and  $AC$  in  $X$  and  $Y$  respectively, so that  $AX : XB = AY : YC$ .

It is reqd. to prove that the str. line  $XY$  is  $\parallel$  to the side  $BC$ .

**Proof.** If the str. line  $XY$  be not  $\parallel$  to  $BC$ , let  $XY'$  be  $\parallel$  to  $BC$  and let it cut the side  $AC$  in  $Y'$ .

$$\therefore AX : XB = AY' : Y'C.$$

But, by hypothesis,  $AX : XB = AY : YC$ .

$$\therefore AY' : Y'C = AY : YC.$$

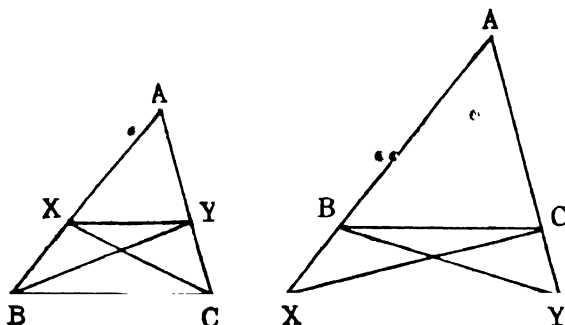
$\therefore$  by Componendo in Fig. 1 and by Dividendo in Fig. 2,

$$AC : Y'C = AC : YC.$$

$\therefore YC = Y'C$  i.e. the two points  $Y, Y'$  coincide with each other.

the str. line  $XY$  is  $\parallel$  to  $BC$ .



**Alternative Method.**

**Construction.** Join  $BY$ ,  $CX$ .

**Proof.** Since the vertex  $Y$  of the  $\triangle AXY$  and the  $\triangle BXY$  is the same, they are of equal altitudes.

$$\therefore \triangle AXY : \triangle BXY = AX : XB.$$

Similarly,  $\triangle AXY : \triangle CXY = AY : YC$ .

But  $AX : XB = AY : YC$ .

$$\therefore \triangle AXY : \triangle BXY = \triangle AXY : \triangle CXY.$$

$$\therefore \triangle BXY = \triangle CXY.$$

But these two  $\triangle$ 's stand on the same base  $XY$  and on the same side of it.  $\therefore$  they must be between the same parallels

*i.e.*  $XY$  is  $\parallel$  to  $BC$ .

**Corollary.** If  $X$ ,  $Y$  be two pts. in the sides  $AB$ ,  $AC$  of a  $\triangle ABC$ , so that  $AB : AX = AC : AY$  (or  $AB : BX = AC : CY$ ), then the str. line  $XY$  is  $\parallel$  to the base  $BC$ .

The proof is similar to that of the Corollary to Theorem 3.

**Exercise II(A)**

1. Three parallel straight lines cut any two transversals proportionally. Prove this.

2. Prove that the straight line which joins the middle points of two sides of a triangle is parallel to the third side.

3. Prove that the straight line drawn through the middle point of one of the sides of a triangle parallel to another side bisects the remaining side.

4. Three straight lines  $OAP$ ,  $OBQ$  and  $OCR$  are such that  $AB$  is parallel to  $PQ$  and  $BC$  to  $QR$ ; show that  $AC$  is parallel  $PR$ .

5.  $ABC$  is a triangle and through  $D$ , a point in  $AB$ ,  $DE$  is drawn parallel to  $BC$  cutting  $AC$  in  $E$ ;  $CF$  is drawn parallel to  $EB$  meeting  $AB$  produced in  $F$ . Prove that  $AB$  is a mean proportional between  $AD$  and  $AF$ .

6. Prove that the line joining the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

[Hints: Produce the oblique sides to meet at a point.]

7. From any point in the base of a triangle, straight lines are drawn parallel to the sides of the triangle; show that the point of intersection of the diagonals of every parallelogram so formed lies on a straight line parallel to the base.

8.  $ABC$  and  $ABD$  are two triangles and  $M$  and  $N$  are the middle points of  $BC$  and  $BD$  respectively; prove that  $MN$  is parallel to the join of the centroid of the triangles.

9. From a point  $E$  in the common base of two triangles  $ACB$ ,  $ADB$  straight lines are drawn parallel to  $AC$ ,  $AD$  meeting  $BC$ ,  $BD$  at  $F$ ,  $G$ ; show that  $FG$  is parallel to  $CD$ .

10.  $E$  is the middle point of the median  $AD$  of a triangle  $ABC$  and  $BE$  is produced to cut  $AC$  in  $F$ ; prove that  $AF = \frac{1}{3}AC$ .

[Hints: Draw  $DG \parallel$  to  $BF$ .]

11.  $OPQ$  is a straight line drawn through a fixed point  $O$  and is such that  $OP : OQ$  is constant. If  $P$  moves along a fixed straight line, find the locus of  $Q$ .

## 2.2. Equiangular Triangles and Polygons.

Two triangles or two polygons are said to be equiangular to each other, if the angles of the one are equal to the angles of the other, when taken in the same order.

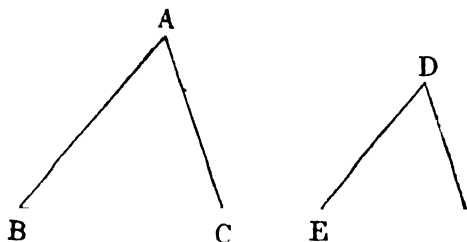
In equiangular triangles the sides opposite to the equal angles are called **corresponding sides**; and in equiangular polygons

the sides adjacent to two successive equal angles are called corresponding sides.

### Similar Triangles and Polygons.

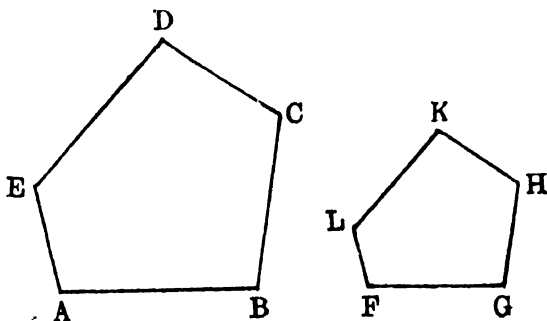
If two equiangular triangles or polygons have their corresponding sides proportional, they are said to be similar.

It will be seen later on that two equiangular triangles are always similar. But two equiangular polygons may or may not be similar.



In the two triangles  $ABC$  and  $DEF$  given above, if  $\angle A = \angle D$ ,  $\angle B = \angle E$ ,  $\angle C = \angle F$  and if  $AB : DE = BC : EF = CA : FD$ , then the two  $\triangle^s ABC, DEF$  will be similar.

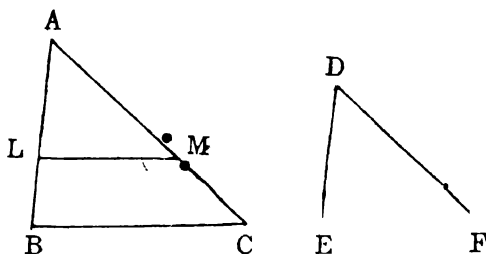
Again in the two polygons  $ABCDE$  and  $FGHKL$  given here, if  $\angle A = \angle F$ ,  $\angle B = \angle G$ ,  $\angle C = \angle H$ ,  $\angle D = \angle K$ ,  $\angle E = \angle L$  and if  $AB : FG = BC : GH = CD : HK = DE : KL = EA : LF$ , then the two polygons  $ABCDE, FGHKL$  are similar. Since  $\angle A = \angle F$  and  $\angle B = \angle G$ , the two sides  $AB$  and  $FG$  being adjacent to these equal angles are corresponding sides of the two polygons; for similar reasons  $BC$  and  $GH$  are corresponding sides, etc.



**N. B.** The sum of the three angles of a triangle being two right angles, if two angles of a triangle be equal to two angles of another, the two triangles will be equiangular.

### Theorem 5

*If two triangles are equiangular, their corresponding sides are proportional.*



In the  $\triangle^s ABC$  and  $DEF$ , let  $\angle A = \angle D$ ,  $\angle B = \angle E$  and  $\angle C = \angle F$ .

It is required to prove that  $AB : DE = BC : EF = AC : FD$ .

**Proof.** From  $AB$ ,  $AC$  cut off  $AL$ ,  $AM$  equal to  $DE$ ,  $DF$  respectively and join  $LM$ .

Then in the  $\triangle^s ALM$  and  $DEF$ ,

$$AL = DE, AM = DF \text{ and } \angle A = \angle D. \quad (\text{Hyp.})$$

$\therefore$  the  $\triangle^s ALM$  and  $DEF$  are equal in all respects.

$\therefore \angle L = \angle E = \angle B$ ,  $\therefore LM$  is  $\parallel$  to  $BC$ .

$\therefore AB : AL = AC : AM$  i.e.  $AB : DE = AC : DF$ .

Similarly, it may be proved that  $BC : EF = AC : DF$ .

$\therefore AB : DE = BC : EF = AC : DF$ .

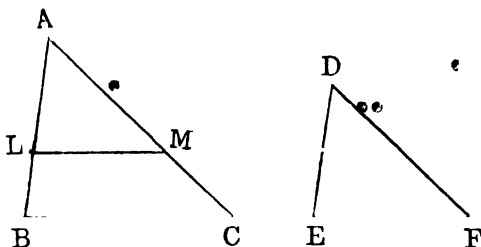
### Theorem 6

[ CONVERSE OF THEOREM 5 ]

*If two triangles have their sides proportional, when taken in order, they are equiangular, equal angles being opposite to corresponding sides.*

Let  $ABC$  and  $DEF$  be two  $\triangle^s$  in which

$$AB : DE = BC : EF = CA : FD.$$



It is reqd. to prove that the  $\angle^s A, B, C$  opposite to the sides  $BC, CA, AB$  are respectively equal to the  $\angle^s D, E, F$  opposite to the corresponding sides  $EF, FD, DE$ .

**Construction.** From  $AB, AC$  cut off  $AL, AM$  equal respectively to  $DE, DF$ . Join  $LM$ .

**Proof.** Since  $AB : DE = AC : DF$  (Hyp.) and, by construction,  
 $DE = AL$  and  $DF = AM$ ,

$$\therefore AB : AL = AC : AM.$$

$$\therefore LM \text{ is } \parallel \text{ to } BC.$$

$\therefore$  the  $\angle ALM =$  the corresponding  $\angle ABC$  and the  $\angle AML =$  the corresponding  $\angle ACB$ .

$\therefore$  the  $\triangle^s ALM$  and  $ABC$  are equiangular.

$\therefore AB : AL = BC : LM$ , that is  $AB : DE = BC : LM$ .

But  $AB : DE = BC : EF$  (Hyp.).  $\therefore LM = EF$ .

And  $AL = DE$  and  $AM = DF$  (by construction).

$\therefore$  the  $\triangle^s ALM$  and  $DEF$  are equal in all respects.

$\therefore$  the  $\angle ALM =$  the  $\angle DEF$  and the  $\angle AML =$  the  $\angle DFE$ .

But the  $\angle ALM =$  the  $\angle B$  and the  $\angle AML =$  the  $\angle C$ .

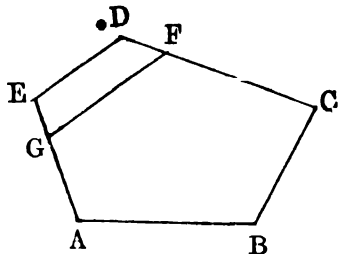
$\therefore$  the  $\angle B =$  the  $\angle E$  and the  $\angle C =$  the  $\angle F$ .

$\therefore$  the remaining  $\angle A =$  the remaining  $\angle D$ .

$\therefore$  the  $\triangle^s ABC$  and  $DEF$  are equiangular.

**N. B.** In Theorem 5, it has been proved that when two triangles are equiangular, they are similar. And it has also been pointed out that two equiangular polygons may not be similar.

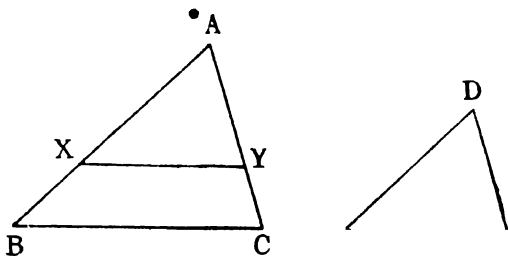
Suppose  $ABCDE$  is a polygon. Through a point  $F$  in  $CD$  draw  $FG \parallel$  to  $DE$  to cut the side  $EA$  in  $G$ . It is clear from the figure that the two polygons  $ABCDE$  and  $ABCF'G$  are equiangular, but their corresponding sides can never be proportional. If they were proportional, then  $BC : CF$  would have been equal to  $BC : CD$  i.e.,  $CF$  would have been equal to  $CD$ , which is impossible.



In order that two polygons may be similar, two conditions are to be satisfied (1) the two polygons should be equiangular and (2) their corresponding sides should also be proportional. But in the case of two triangles, one of the two conditions will be sufficient to make them similar. Hence, if two triangles are equiangular or their corresponding sides are proportional, they will be similar.

### Theorem 7

*If two triangles have one angle of the one equal to one angle of the other and the sides about the equal angles proportional, the triangles are similar.*



In the  $\triangle ABC$  and  $DEF$ , let the  $\angle A = \angle D$  and  $AB : DE = AC : DF$ .

It is required to prove that  $\triangle ABC$  and  $\triangle DEF$  are similar.

**Construction.** From the sides  $AB$  and  $AC$  of the  $\triangle ABC$  cut off  $AX$  and  $AY$  respectively equal to  $DE$  and  $DF$ . Join  $XY$ .

**Proof.** In the  $\triangle AXY$  and  $\triangle DEF$ ,  $AX=DE$ ,  $AY=DF$  and the included  $\angle A$  = the included  $\angle D$ .

$\therefore$  the  $\triangle AXY$  and  $\triangle DEF$  are equal in all respects.

$\therefore \angle AXY = \angle E$  and  $\angle AYX = \angle F$ .

Now, since  $AB : DE = AC : DF$ .  $\therefore AB : AX = AC : AY$ .

$\therefore XY$  is  $\parallel$  to  $BC$ .

$\therefore$  the  $\angle AXY$  = the corresponding  $\angle B$  and the  $\angle AYX$  = the corresponding  $\angle C$ .

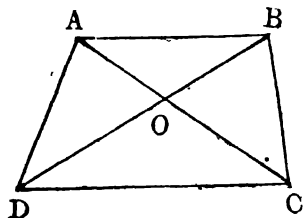
But the  $\angle AXY$  = the  $\angle E$  and the  $\angle AYX$  = the  $\angle F$ .

$\therefore$  the  $\angle B$  = the  $\angle E$  and the  $\angle C$  = the  $\angle F$ .

the  $\triangle ABC$  and  $\triangle DEF$  are equiangular, and hence similar.

### 2.3. Illustrative Examples

**Ex. 1.**  $AB$  and  $DC$  are the parallel sides of a trapezium whose diagonals  $AC$  and  $BD$  intersect at  $O$ ; prove that  $AB : CD = OA : OC = OB : OD$ .



Let the diagonals  $AC$ ,  $BD$  of the trapezium with the sides  $AB$ ,  $DC$  parallel intersect at  $O$ .

It is reqd. to prove that

$$AB : CD = OA : OC = OB : OD.$$

**Proof.** In the  $\triangle AOB$ ,  $\triangle COD$ ,  $\angle OAB = \angle OCD$  and  $\angle OBA = \angle ODC$ .

[  $\because AB$  is  $\parallel$  to  $DC$  ]

$\therefore$  the  $\triangle AOB$ ,  $\triangle COD$  are equiangular, and hence similar.

$\therefore AB : CD = OA : OC = OB : OD$ .

**Ex. 2.** Prove that a line drawn parallel to the parallel sides of a trapezium through the point of intersection of the diagonals is bisected at the point.

Let the diagonals  $AC$ ,  $BD$  of the trapezium  $ABCD$  intersect at  $O$ . Let the straight line  $POQ$  drawn through  $O$  parallel to  $AB$  or  $CD$  cut the sides  $AD$ ,  $BC$  at  $P$ ,  $Q$  respectively.

It is required to prove that  $OP = OQ$ .

**Proof.** In the  $\triangle OAB$  and  $OCD$ ,  
 $\angle OAB = \angle OCD$  and  $\angle OBA = \angle ODC$

[  $\because AB$  is  $\parallel$  to  $DC$  ]

$\therefore$  the  $\triangle OAB$  and  $OCD$  are equiangular and hence similar.

$\therefore OA : OC = OB : OD$ .

$\therefore (OA + OC) : OC = (OB + OD) : OD$ .

[ by Componendo ]

i.e.  $AC : OC = BD : OD$ .

Again, since  $OP$  is  $\parallel$  to  $AB$ , the  $\triangle DPO$  and  $DAB$  are similar.

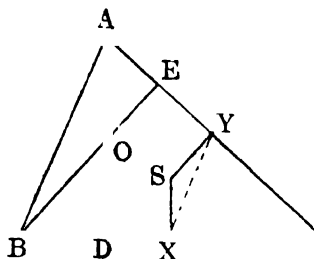
$\therefore BD : OD = AB : OP$ .

In a similar way,  $AC : OC = AB : OQ$ .

But  $AC : OC = BD : OD$  (proved),

$\therefore AB : OP = AB : OQ$ ; that is,  $OP = OQ$ .

**Ex. 3.** In any triangle, prove that the distance of the ortho-centre from any angular point is double the distance of the circum-centre from the opposite side.



Let the pt. of intersection of  $AD$ ,  $BE$ , perpendiculars drawn to  $BC$ ,  $CA$  respectively, be  $O$ , the ortho-centre and the pt. of intersection of the perps. drawn to  $BC$ ,  $CA$  from their middle points  $X$ ,  $Y$  be  $S$ , the circum-centre of the  $\triangle ABC$

It is reqd. to prove that  $OA = 2SX$ .

**Construction.** Join  $XY$ .

**Proof.** Since  $X$ ,  $Y$  are the middle points of  $BC$ ,  $CA$  respectively,

$\therefore XY$  is  $\parallel$  to  $AB$  and  $= \frac{1}{2}AB$ .

Now,  $SX$  is  $\parallel$  to  $AD$ , both being perps. to  $BC$ ,

and  $SY$  is  $\parallel$  to  $BE$ , both being perps. to  $CA$ .

$\therefore$  the two  $\triangle SXY$  and  $OAB$  are equiangular, and hence similar.

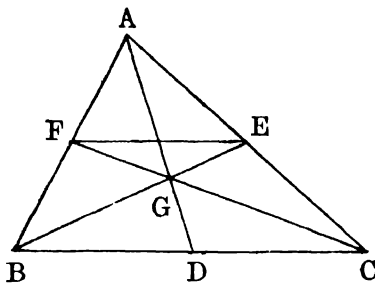
$\therefore OA : SX = AB : XY = 2 : 1$ .

i.e.  $OA = 2SX$ .

**Ex. 4.** Prove that the two medians of a triangle intersect at their point of trisection. Hence, show that the medians of a triangle are concurrent.



Let  $E$  and  $F$  be the middle points of the sides  $AC$  and  $AB$  of the  $\triangle ABC$  and let the medians  $BE$  and  $CF$  intersect at  $G$  and let  $D$  be the middle point of the side  $BC$ .



It is required to prove that  
 (i)  $GE = \frac{1}{3}BE$  and  $GF = \frac{1}{3}CF$  and  
 (ii) the three medians of the  $\triangle ABC$  are concurrent.

**Construction.** Join  $EF$ .

**Proof.** Since  $E$  and  $F$  are middle points of  $AC$  and  $AB$ ,  $FE \parallel$  to  $BC$  and  $= \frac{1}{2}BC$  and,  $\therefore$  the  $\triangle FGE$  and  $BGC$  are equiangular, and hence similar.

$$\therefore BG : GE = CG : GE = BC : FE = 2 : 1.$$

$$\therefore GE = \frac{1}{3}BE \text{ and } GF = \frac{1}{3}CF.$$

In a similar way, if the medians  $AD$  and  $BE$  intersect at  $G'$ , it may be shown that  $G'E = \frac{1}{3}BE$ , i.e. the two pts.  $G$  and  $G'$  are coincidental and  $G'D = \frac{1}{3}AD$  i.e.,  $GD = \frac{1}{3}AD$ .

**Ex. 5.** The diagonal  $BD$  of a parallelogram  $ABCD$  is divided at  $E$  so that  $BE$  is one-third of  $ED$  and  $AE$ ,  $DC$  is produced to meet at  $F$ ; show that  $FC = 2AB$ .

Let  $E$  be a point on the diagonal  $BD$  of the parallelogram  $ABCD$  such that  $BE = \frac{1}{3}ED$ ; let the join of  $A$  and  $E$  be produced to meet  $DC$  produced in  $F$ .

It is reqd. to prove that  $FC = 2AB$ .

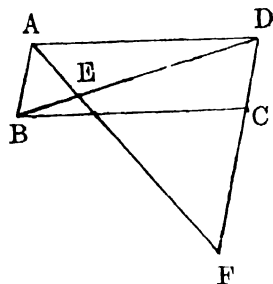
**Proof.** Since  $AB$  and  $DF$  are parallel, the two  $\triangle ABE$  and  $DFE$  are equiangular, and hence similar.

$$\therefore AB : DF = BE : ED = 1 : 3.$$

$$\therefore DF = 3AB.$$

But  $AB = DC$ , being the opposite sides of a parallelogram.

$$\therefore FC = 2AB.$$



**Ex. 6.**  $ABC$  is an isosceles triangle having each of the angles at the base double of the vertical angle  $BAC$ ; the bisector of the angle  $ACB$  meets  $AB$  at  $D$ . Show that  $AB, BC, BD$  are proportionals.

Let  $ABC$  be an isosceles  $\triangle$  in which  $AB=AC$  and each of the angles  $ABC$  and  $ACB$  at the base is double of the vertical  $\angle BAC$ ; let  $CD$ , the bisector of the  $\angle ACB$  meet  $AB$  at  $D$ .

It is reqd. to prove that  $AB$ ,  $BC$  and  $BD$  are in continued proportion i.e.  $AB : BC = BC : BD$ .

**Proof.** Since the  $\angle ABC =$  the  $\angle ACB = 2\angle BAC$ , and the sum of the  $\angle$ 's of the  $\triangle ABC = 180^\circ$ ,

$\therefore$  the  $\angle ABC =$  the  $\angle ACB = 72^\circ$  and the  $\angle BAC = 36^\circ$ .

Again,  $CD$  is the bisector of the  $\angle ACB$ ,  
 $\therefore$  the  $\angle BCD = 36^\circ$ .

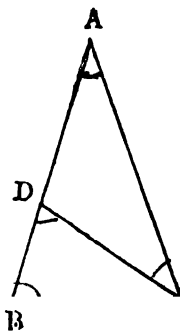
$\therefore$  in the  $\triangle BCD$ , the  $\angle BCD = 36^\circ$  and the  $\angle DBC$  i.e.  $\angle ABC = 72^\circ$ .

$\therefore$  the remaining  $\angle BDC = 72^\circ$ .

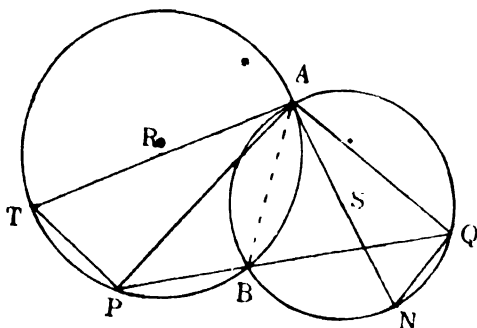
$\therefore$  the  $\triangle ABC$  and  $BCD$  are equiangular, and hence similar.

$\therefore AB : BC = BC : BD$ .

i.e.  $AB$ ,  $BC$  are in continued proportion.



**Ex. 7.** Two circles intersect at  $A$  and  $B$ , and through  $B$  any straight line  $PBQ$  is drawn to cut the two circles at  $P$  and  $Q$ . Prove that their diameters are as  $AP : AQ$ .



Let the two circles whose centres are at  $R$  and  $S$  intersect at  $A$  and  $B$ , and through  $B$  let  $PBQ$  be drawn to cut the two circles at  $P$  and  $Q$ . Join  $AP$  and  $AQ$ ; let  $ART$  and  $ASN$  be diameters of the two circles.

It is reqd. to prove that  $AT : AN = AP : AQ$ .

**Proof.** Join  $AB$ ,  $TP$  and  $NQ$ .

Now, the  $\angle ANQ =$  the  $\angle ABQ$ , ( being in the same segment )

Again,  $ABPT$  is a cyclic quadrilateral,  $\therefore$  the ext.  $\angle ABQ =$  the opposite int.  $\angle ATP$ .  $\therefore$  the  $\angle ANQ =$  the  $\angle ATP$ .

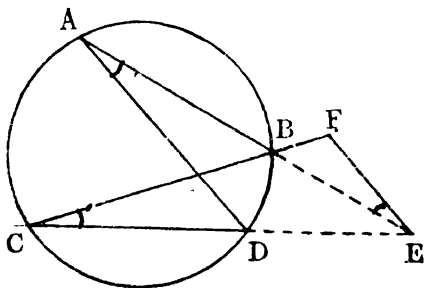
In the  $\triangle APT$  and  $AQN$ ,

the  $\angle ATP =$  the  $\angle ANQ$  ( proved ) and the  $\angle APT =$  the  $\angle AQN$  (being in semi-circles).

$\therefore$  the triangles are equiangular, and hence similar.

$$\therefore AT : AN = AP : AQ.$$

**Ex. 8.** Chords  $AB$  and  $CD$  of a circle are produced to meet at  $E$ , and  $EF$  is drawn through  $E$  parallel to  $AD$  meeting  $CB$  produced in  $F$ . Prove that  $EF$  is a mean proportional between  $FB$  and  $FC$ .



Let the two chords  $AB$ ,  $CD$  of a circle be produced to meet at  $E$  and through  $E$  let  $EF$  be drawn  $\parallel$  to  $AD$  to meet  $CB$  produced at  $F$ .

It is reqd. to prove that  $EF$  is a mean proportional between  $FB$  and  $FC$ .

**Proof.** In the  $\triangle EFB$  and  $EFC$ , the  $\angle FEB$  is common and the  $\angle ECF =$  the  $\angle BAD$  ( in the same segment )  
 $=$  the  $\angle BEF$  [  $\because EF$  is  $\parallel$  to  $AD$  ]

$\therefore$  the two  $\triangle EFB$  and  $EFC$  are equiangular, and hence similar.

$$\therefore FB : FE = FE : FC.$$

i.e.  $FE$  is a mean proportional between  $FB$  and  $FC$ .

**Ex. 9.** A common tangent to two circles cuts their line of centres internally or externally in the ratio of their radii. Prove this.

Let  $O$  and  $O'$  be the centres of the two circles and  $PT$  their transverse common tangent in Fig. 1 and direct common tangent in Fig. 2 and let the tangent cut the line of centres (i) in Fig. 1, and (ii)  $OO'$  produced in Fig 2, at  $R$ .

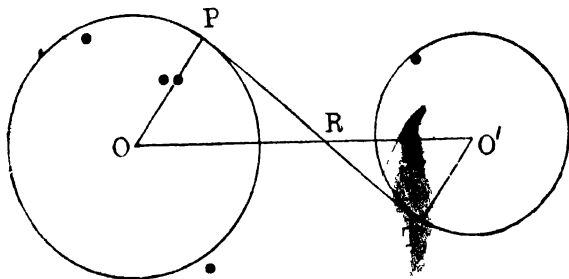


Fig. 1

It is required to prove (in both Figs.) that  $OR : O'R = OP : O'T$ .

**Proof.** Join  $OP$  and  $O'T$ .

In the  $\triangle OPR$  and  $O'TR$ , the  $\angle OPR = \angle O'TR$  (the radius through the point of contact being perp. to the tangent, each is a rt.  $\angle$ ) and the  $\angle ORP = \angle O'TR$  (vertically opp.  $\angle$ 's in Fig. 1, and the same  $\angle$  in Fig. 2.)

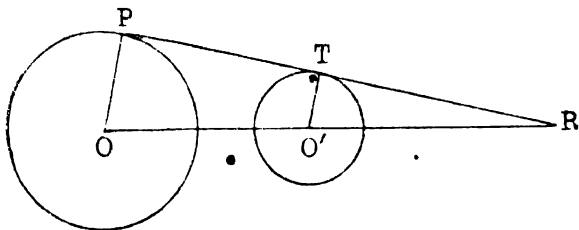


Fig. 2

$\therefore$  the  $\triangle OPR$  and  $O'TR$  are equiangular, and hence similar.

$\therefore OR : O'R = OP : O'T$ .

**Ex. 10.** If the internal bisector of the vertical angle of a triangle meets the base, prove that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the bisector.

In the  $\triangle ABC$  let  $AD$ , the bisector of the vertical  $\angle BAC$  meet the base  $BC$  at  $D$ .

It is reqd. to prove that

$$AB \cdot AC = BD \cdot DC + AD^2.$$

**Construction.** Draw the circum-circle of the  $\triangle ABC$  and produce  $AD$  to meet the circum-circle in  $P$ .

**Proof.** Join  $BP$ .

In the  $\triangle ADC$  and  $BDP$ , the  $\angle CAD$  = the  $\angle PBD$  and the  $\angle ACD$  = the  $\angle BPD$  (in the same segment).

$\therefore$  the  $\triangle ADC$  and  $BDP$  being equiangular are similar.

$$\therefore BD : DP = AD : DC \text{ i.e., } AD \cdot DP = BD \cdot DC.$$

Now, in the  $\triangle BAP$  and  $DAC$ , the  $\angle BAP$  = the  $\angle DAC$  (Hyp.)

and the  $\angle BPA$  = the  $\angle DCA$  (in the same segment).

$\therefore$  the  $\triangle BAP$  and  $DAC$  being equiangular are similar.

$$\therefore AB : AP = AD : AC.$$

$$\begin{aligned} \therefore AB \cdot AC &= AP \cdot AD = (AD + DP)AD = AD^2 + AD \cdot DP \\ &= AD^2 + BD \cdot DC. \end{aligned}$$

**Ex. 11.** If from the vertical angle of a triangle a perpendicular is drawn to the base, prove that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle. [Brahmagupta's Theorem]

Let  $ABPC$  be the circum-circle of the  $\triangle ABC$ ,  $AP$  the diameter of the circum-circle and  $AD$  the perpendicular drawn from  $A$  upon the base  $BC$ .

It is reqd. to prove that

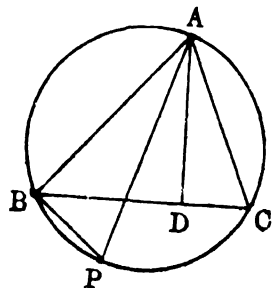
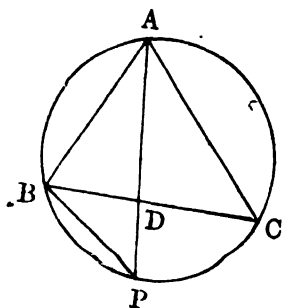
$$AB \cdot AC = AD \cdot AP.$$

**Proof.** Join  $BP$ .

In the  $\triangle BAP$  and  $DAC$ , the  $\angle BPA$  = the  $\angle DCA$  (in the same segment) and the  $\angle ABP$  = the  $\angle ADC$  [ $\because AD$  is  $\perp BC$  and  $\angle ABP$  in a semi-circle = a rt.  $\angle$ .]

$\therefore$  the  $\triangle BAP$  and  $DAC$  are equiangular, and hence similar.

$$\therefore AB : AD = AP : AC. \therefore AB \cdot AC = AD \cdot AP.$$



**N. B.** With the help of Brahmagupta's Theorem a useful Trigonometrical formula can be established.

If the sides of triangle  $ABC$  opposite to the  $\angle A$ , the  $\angle B$  and the  $\angle C$  be denoted by  $a$ ,  $b$  and  $c$  respectively, the radius of its circum-circle by  $R$  and its area by  $S$ , then by the above Theorem it can be easily proved that

$$R = \frac{abc}{4S},$$

From Brahmagupta's Theorem, we get  $AB \cdot AC = AD \cdot AP$ .

$$\text{i.e., } c \cdot b = AD \cdot 2R.$$

$$\therefore 2R = \frac{bc}{AD} = \frac{bc \cdot a}{AD \cdot a} = \frac{abc}{2S} \quad [\because S = \frac{1}{2}a \cdot AD] \therefore R = \frac{abc}{4S}.$$

**Ex. 12.** Prove that the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the two rectangles contained by its two opposite sides. [Ptolemy's Theorem]

Let  $ABCD$  be a cyclic quadrilateral and  $AC$ ,  $BD$  its diagonals.

It is reqd. to prove that

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

**Construction.** Make the  $\angle DAX$  equal to the  $\angle BAC$  and let  $AX$  meet the diagonal  $BD$  in  $X$ .

**Proof.** The  $\angle BAC = \angle DAX$ .

(by construction)

To each of these add the  $\angle CAX$ .

$$\therefore \text{the } \angle BAX = \text{the } \angle CAD.$$

Now, in the  $\triangle BAX$  and  $ACD$ , the  $\angle BAX = \angle CAD$  (proved) and the  $\angle ABX = \angle ACD$  (in the same segment).

$\therefore$  the  $\triangle BAX$  and  $ACD$ , being equiangular, are similar.

$$\therefore AB : AC = BX : CD.$$

$$\therefore AB \cdot CD = AC \cdot BX \quad (\text{by cross-multiplication}) \quad \dots \quad (1)$$

Again, the  $\angle BAC = \angle DAX$  (by construction)

and the  $\angle ACB = \angle ADB$  i.e.,  $\angle ADC$  (in the same segment).

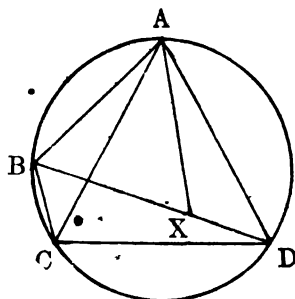
$\therefore$  the two  $\triangle ABC$  and  $AXD$  are similar.

$$\therefore BC : XD = AC : AD.$$

$$\therefore BC \cdot AD = AC \cdot XD \quad (\text{by cross-multiplication}) \quad \dots \quad (2)$$

Now by adding (1) and (2), we have

$$\begin{aligned} AB \cdot CD + BC \cdot AD &= AC \cdot BX + AC \cdot XD \\ &= AC(BX + XD) = AC \cdot BD. \end{aligned}$$



## Exercise II(B)

1. In a triangle  $ABC$  the bisector of the vertical angle  $BAC$  meets the base at  $D$  and the circum-circle at  $E$ ; show that  $AB$ ,  $AD$ ,  $AE$  and  $AC$  are four proportionals.

2. Prove that the straight line joining the middle points of the parallel sides of a trapezium passes through the point of intersection of the diagonals.

3. Prove that the point of intersection of the oblique sides of a trapezium, the middle points of the parallel sides and the point of intersection of the diagonals are concurrent.

4. Through the point of contact  $A$  of two circles which touch each other, a straight line is drawn to cut the two circles at  $P$  and  $Q$ . Prove that their diameters are as  $AP : AQ$ .

5. (i) Prove that the median of a triangle bisects all straight lines parallel to the side the median bisects.

(ii) In similar triangles show that the corresponding medians make equal angles with the corresponding sides.

6. Two straight lines  $AB$  and  $CD$  intersect at  $O$ , internally, or externally, so that  $OA : OC = OD : OB$ . Show that the four points  $A, C, B, D$  are concyclic.

7. If two triangles are on equal bases and between the same parallels, prove that the intercepts made by the sides of the triangle on any line parallel to the line of the bases are equal.

8.  $ABC$  is a right-angled triangle, and from  $A$ , a perpendicular  $AD$  is drawn to the hypotenuse  $BC$ ; prove that

$$BD : DC = AB^2 : AC^2.$$

9. From the angle  $A$  of a parallelogram  $ABCD$  a straight line is drawn cutting the diagonal  $BD$  in  $P$ , and the sides  $BC$  and  $DC$  (produced where necessary) in  $Q$  and  $R$  respectively; prove that  $AP$  is a mean proportional between  $PQ$  and  $PR$ .

10. (i) In a triangle  $ABC$ , the straight line  $DEF$  meets the sides  $BC, CA, AB$  at the points  $D, E, F$  respectively, and makes equal angles with  $AB$  and  $AC$ ; prove that  $BD : CD = BF : CE$ .

(ii) In a triangle  $ABC$ ,  $F$  and  $E$  are points on  $AB$  and  $AC$  such that  $BF = CE$ . If  $FE$  and  $BC$ , when produced, meet at  $D$ , prove that  $AB : AC = DE : DF$ .

11. If two circles touch externally, prove that their common tangent is a mean proportional between their diameters.

12. In two circles any two parallel radii are drawn, one in each circle; prove that the straight line joining their extremities cuts the line of centres in one or other of two fixed points.

13.  $ABC$  is a triangle and  $X$  is any point in  $BC$ ; prove that the circum-radii of the triangles  $ABX$  and  $ACX$  are as  $AB : AC$ .

14. If the diagonals of a cyclic quadrilateral are at right angles, prove that the sum of the rectangles contained by its opposite sides is equal to twice the area of the quadrilateral.

15.  $AC$  and  $BD$  are perpendiculars on  $CD$ , and  $AD$ ,  $BC$  intersect at  $P$ . If  $PQ$  be perpendicular to  $CD$ , show that  $PQ$  bisects the angle  $AQB$ .

16. (i)  $P$  is the middle point of the median  $AD$  of the triangle  $ABC$ . If  $BP$  produced meet  $CA$  at  $E$ , prove that  $CE = 2AE$ .

(ii)  $P$  is any point on the median  $AD$  of the triangle  $ABC$ . If  $BP$  and  $CP$ , when produced, meet  $AC$  and  $AB$  at  $E$  and  $F$ , prove that  $FE$  is parallel to  $BC$ .

### Theorem 8

*The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing that angle, and likewise the external bisector externally.*

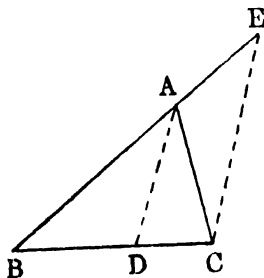


Fig. 1

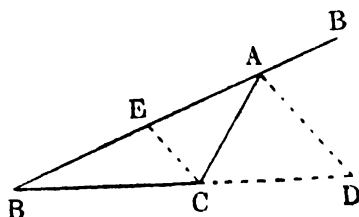


Fig. 2



Let  $AD$ , the bisector of the  $\angle BAC$  of the  $\triangle ABC$  divide the side  $BC$  internally in Fig. 1 at the pt.  $D$  and in Fig. 2, let the str. line  $AD$  bisect the exterior  $\angle B'AC$  and cut  $BC$  produced at  $D$ .

It is reqd. to prove, in both cases, that  $BD : CD = AB : AC$ .

**Construction.** Through  $C$  draw  $CE \parallel$  to  $DA$  to meet (i)  $BA$  produced at  $E$  (Fig. 1) or (ii)  $BA$  at  $E$  (Fig. 2).

**Proof.** Since  $AD$  is  $\parallel$  to  $EC$ ,  $\therefore$  the  $\angle DAC =$  the alt.  $\angle ACE$  and the  $\angle BAD$  or the  $\angle B'AD =$  corresponding  $\angle AEC$ .

But the  $\angle BAD$  or  $\angle B'AD =$  the  $\angle DAC$  (Hyp.).

$\therefore$  the  $\angle ACE =$  the  $\angle AEC$ ,  $\therefore AC = AE$ .

Now,  $AD$  is  $\parallel$  to  $EC$ ,  $\therefore BD : DC = BA : AE = BA : AC$ .

**N. B.** The converse of this Theorem is also true and this is shown as follows.

### Converse Theorem.

If a straight line drawn from an angle of a triangle divides the opposite side (i) internally or (ii) externally in the ratio of the sides containing the angle, the straight line is the (i) internal or (ii) external bisector of that angle.

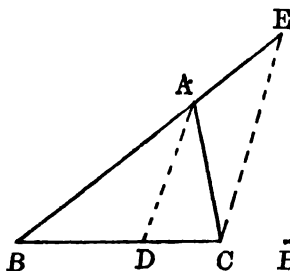


Fig. 1

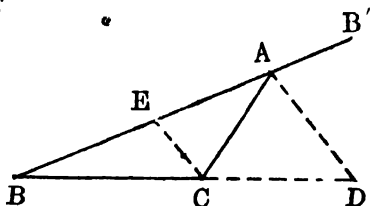


Fig. 2

In the  $\triangle ABC$  let the str. line  $AD$  drawn from the  $\angle A$  divide  $BC$  (in Fig. 1) or  $BC$  produced (in Fig. 2) at  $D$ , so that  $BD : DC = BA : AC$ .

It is reqd. to prove that  $AD$  is the bisector of the  $\angle BAC$  (in Fig. 1) or of the external  $\angle B'AC$  (in Fig. 2).

**Construction.** Through  $C$  draw  $CE \parallel$  to  $DA$  to meet  $BA$  produced at  $E$  (in Fig. 1) or  $BA$  at  $E$  (in Fig. 2).

**Proof.** Since  $AD$  is  $\parallel$  to  $EC$ ,  $\therefore BD : DC = BA : AE$ .

But  $BD : DC = BA : AC$  (Hyp.).  $\therefore AE = AC$ .

$\therefore$  the  $\angle AEC =$  the  $\angle ACE$ .

Again, since  $AD$  and  $EC$  are  $\parallel$ , the  $\angle CAD =$  the alt.  $\angle ACE$  and in Fig. 1, the  $\angle BAD$  or in Fig. 2, the  $\angle B'AD =$  the corresponding  $\angle AEC$ .

But the  $\angle AEC =$  the  $\angle ACE$ .

$\therefore$  the  $\angle BAD$  or  $\angle B'AD =$  the  $\angle CAD$ .

That is, the line  $AD$  bisects (i) the  $\angle BAC$  (in Fig. 1) or (ii) the ext.  $\angle B'AC$  (in Fig. 2).

### Theorem 9

*In a right-angled triangle if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.*

In the  $\triangle ABC$  let the  $\angle A$  be a rt.  $\angle$  and let  $AD$  be drawn perpendicular to  $BC$  from the angular pt.  $A$ .

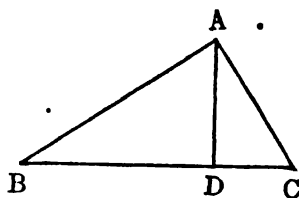
It is reqd. to prove that the  $\triangle^s ADB$ ,  $ADC$  and  $ABC$  are similar.

**Proof.** In the  $\triangle^s ADC$  and  $ABC$ , the  $\angle ADC =$  the  $\angle BAC$  (being rt.  $\angle^s$ ) and the  $\angle C$  is common.

$\therefore$  the  $\triangle^s ADC$  and  $ABC$  are equiangular, and hence similar. Similarly, it may be proved that the  $\triangle^s ADB$  and  $ABC$  are similar.

$\therefore$  the  $\triangle^s ADB$  and  $ADC$ , both being equiangular to the  $\triangle ABC$ , are equiangular to one another.

$\therefore$  the  $\triangle^s ADC$  and  $ADB$  are similar.



**Cor. 1.**  $\triangle^s ADC$  and  $ADB$  are similar,

$$\therefore CD : AD = AD : BD.$$

$$\therefore AD^2 = BD \cdot CD.$$

i.e.,  $AD$  is a mean proportional between  $BD$  and  $CD$ .

**Cor. 2.**  $\triangle^s ADB$  and  $ABC$  are similar,

$$\therefore BD : AB = AB : BC.$$

$$\therefore AB^2 = BD \cdot BC.$$

i.e.,  $AB$  is a mean proportional between  $BD$  and  $BC$ .

**Cor. 3.**  $\triangle^s ADC$  and  $ABC$  are similar,

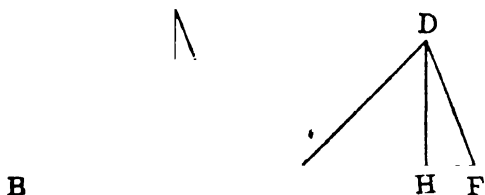
$$\therefore CD : AC = AC : BC.$$

$$\therefore AC^2 = CD \cdot BC.$$

i.e.,  $AC$  is a mean proportional between  $CD$  and  $BC$ .

### Theorem 10

*The ratio of the areas of similar triangles is equal to the ratio of the squares on their corresponding sides.*



Let the two similar  $\triangle^s ABC, DEF$  have the  $\angle A = \text{the } \angle D$ , the  $\angle B = \text{the } \angle E$  and the  $\angle C = \text{the } \angle F$ . Therefore,  $BC$  and  $EF$  are corresponding sides.

It is reqd. to prove that  $\triangle ABC : DEF = BC^2 : EF^2$ .

**Construction.** From  $A$  and  $D$  draw  $AG$  and  $DH$  perps. to  $BC$  and  $EF$  respectively.

**Proof.** In the  $\triangle^s ABG$  and  $DEH$ , the  $\angle B = \text{the } \angle E$  (Hyp.)

and the  $\angle AGD =$  the  $\angle DHE$  (being rt.  $\angle^s$ )

$\therefore$  the  $\triangle^s ABG$  and  $DEH$  are equiangular, and hence similar.

$$\therefore AG : DH = AB : DE = BC : EF$$

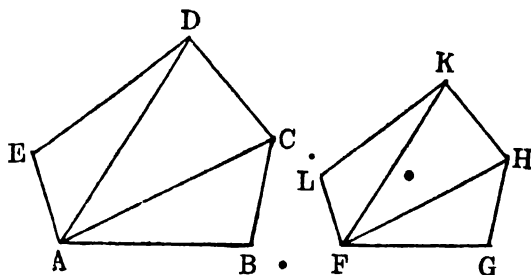
[  $\because$  the  $\triangle^s ABC$  and  $DEF$  are similar ]

Again,  $\triangle ABC = \frac{1}{2} BC \cdot AG$  and  $\triangle DEF = \frac{1}{2} EF \cdot DH$ .

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} BC \cdot AG}{\frac{1}{2} EF \cdot DH} = \frac{BC \cdot AG}{EF \cdot DH} = \frac{BC \cdot BC}{EF \cdot EF} = \frac{BC^2}{EF^2}.$$

[  $\because$  the  $\triangle^s$  are similar. ]

**Cor.** Areas of two similar polygons are to one another as the squares on their corresponding sides.



It may be easily proved with the help of Theorem 7 that if two corresponding vertices  $A, F$  of two similar polygons  $ABCDE, FGHLK$  be joined to the other vertices of the respective polygons, the two polygons will be divided into an equal number of similar triangles.

$\therefore$  the  $\triangle^s ABC$  and  $FGH$ , the  $\triangle^s ACD$  and  $FHK$  and the  $\triangle^s ADE$  and  $FKL$  are similar.

$$\therefore \frac{\triangle ABC}{\triangle FGH} = \frac{AC^2}{FH^2} = \frac{\triangle ACD}{\triangle FHK} = \frac{AD^2}{FK^2} = \frac{\triangle ADE}{\triangle FKL}.$$

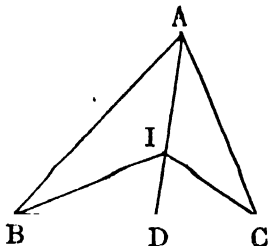
$$\therefore \frac{\triangle ABC}{\triangle FGH} = \frac{\triangle ACD}{\triangle FHK} = \frac{\triangle ADE}{\triangle FKL}.$$

$$\begin{aligned}\therefore \frac{\triangle ABC}{\triangle FGH} &= \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle FGH + \triangle FHK + \triangle FKL} \quad (\text{by Addendo}) \\ &= \frac{\text{polygon } ABCDE}{\text{polygon } FGHKL}.\end{aligned}$$

$$\text{But } \frac{\triangle ABC}{\triangle FGH} = \frac{AB^2}{FG^2} \quad \therefore \frac{\text{polygon } ABCDE}{\text{polygon } FGHKL} = \frac{AB^2}{FG^2}$$

## 2.4. Illustrative Examples.

**Ex. 1.** Prove, with the help of Theorem 8, that the internal bisectors of the angles of a triangle are concurrent.



Let the str. lines  $BI$  and  $CI$ , the bisectors of the  $\angle ABC$  and  $\angle ACB$  of the  $\triangle ABC$ , intersect each other at the pt.  $I$ . Join  $AI$ .

It is reqd. to prove that the line  $AI$  bisects the  $\angle BAC$ .

**Construction.** Produce  $AI$  to meet  $BC$  in  $D$ .

**Proof.** In the  $\triangle ABD$ , the line  $BI$  bisects the  $\angle ABD$  (Hyp.)

$$\therefore AB : BD = AI : ID.$$

Again, in the  $\triangle ACD$ , the line  $CI$  bisects the  $\angle ACD$  (Hyp.)

$$\therefore AC : CD = AI : ID.$$

$$\therefore AB : BD = AC : CD \quad (\text{each being equal to } AI : ID)$$

$$\therefore AB : AC = BD : CD \quad (\text{by Alternando}).$$

$$\therefore \text{the line } AID \text{ bisects the } \angle BAC. \quad [\text{Theo. 8, Converse}]$$

Hence bisectors of the angles of a triangle are concurrent.

**Ex. 2.** Prove that the locus of a point which moves so that the ratio of its distances from two fixed points is constant, is a circle.

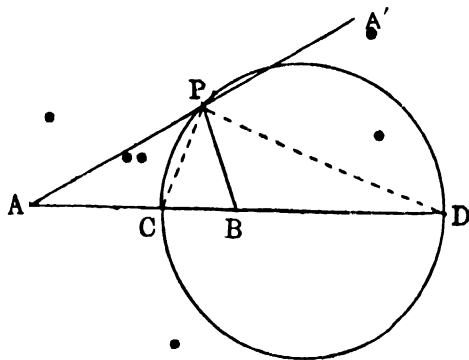
[ Circle of Apollonius ]

Let  $A, B$  be two fixed points and  $P$  a moving point such that, for all positions of  $P$ ,  $PA : PB = a$  constant.

It is reqd. to find the locus of  $P$ .

**Construction.** Let  $P$  be one position of the moving point. Join  $AP, BP$  and produce  $AP$  to  $A'$ . Let the lines  $PC, PD$  bisect the  $\angle APB, A'PB$  respectively and let them meet  $AB$  and  $AB$  produced at the points  $C, D$ .

The circle described on  $CD$  as diameter will be the reqd. locus.



**Proof.** Since  $PC$  bisects the  $\angle APB$ ;

$\therefore AC : BC = PA : PB = \text{a constant.}$

$\therefore C$  is a fixed point. [ $\because A, B$  are fixed pts.]

Similarly, it may be proved that  $D$  also is a fixed pt.

$\therefore$  the position of  $CD$  is fixed.

Again, since  $PC, PD$  are the bisectors of the adjacent  $\angle APB, A'PB$ , the  $\angle CPD = \text{a rt. } \angle$ .

$\therefore$  the fixed str. line  $CD$  always subtends a rt.  $\angle$  at  $P$ .

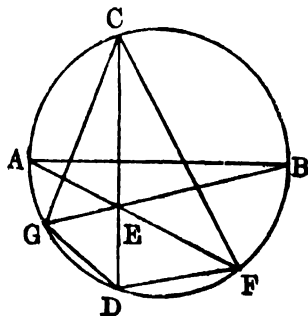
**N. B.** This circle is called the **Circle of Apollonius**.

**Ex. 3.**  $AB$  is a diameter of a circle,  $CD$  is a chord at right angles to it and  $E$  any point in  $CD$ ;  $AE$  and  $BE$  are drawn and produced to cut the circle in  $F$  and  $G$ ; show that the quadrilateral  $CFDG$  has any two of its adjacent sides in the same ratio as the remaining two.

Let  $AB$  be a diameter of the circle and  $CD$  a chord at right angles to  $AB$ .

Let  $E$  be a pt. in  $CD$  and let  $AE, BE$  be produced to cut the circle at  $F, G$  respectively.

It is reqd. to prove that the quadrilateral  $CFDG$  has any two of its adjacent sides in the same ratio as the remaining two.



**Proof.** Since the diameter  $AB$  is perp. to chord  $CD$ ,  $AB$  bisects  $CD$  at rt.  $\angle$ .

$\therefore$  the arc  $BC$  = the arc  $BD$  and the arc  $AC$  = the arc  $AD$ .

$\therefore$  the  $\angle BGC$  = the  $\angle BGD$  and the  $\angle AFC$  = the  $\angle AFD$

(Standing on equal arcs)

$\therefore$  the lines  $GE$  and  $FE$  bisect the  $\angle$ 's  $CGD$  and  $CFD$  respectively.

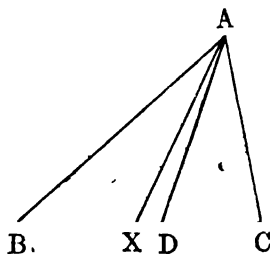
$\therefore CG : GD = CE : ED$  and  $CF : FD = CE : ED$ .

$\therefore CG : GD = CF : FD$ .

$\therefore CG : CF = GD : FD$  (by Alternando).

i.e. any two sides of the quadrilateral  $CFDG$  have the same ratio as the remaining two.

**Ex. 4.** The angle  $A$  of a triangle  $ABC$  is bisected by  $AD$  meeting  $BC$  in  $D$  and  $AX$  is the median bisecting  $BC$ ; show that  $XD$  has the same ratio to  $XB$  as the difference of the sides has to their sum.



Let the bisector  $AD$  of the  $\angle A$  of the  $\triangle ABC$  meet the side  $BC$  at  $D$  and let  $AX$  the median bisect the side  $BC$  at  $X$ .

It is reqd. to prove that  $XD : XB = (AB - AC) : (AB + AC)$ .

**Proof.** Since  $AD$  bisects the  $\angle BAC$ ,

$$\frac{AB}{AC} = \frac{BD}{CD}$$

$$\therefore \frac{AB + AC}{AB - AC} = \frac{BD + CD}{BD - CD} \quad (\text{by Componendo-Dividendo})$$

$$= \frac{BC}{(BX + XD) - (CX - XD)} = \frac{2XD}{2XB} \quad [\because BX = CX]$$

$$= \frac{XD}{XB}$$

$$\therefore \frac{XD}{XB} = \frac{AB - AC}{AB + AC} \quad [\text{by Invertendo}]$$

i.e.  $XD : XB = (AB - AC) : (AB + AC)$ .

**Ex. 5.** In a right-angled triangle, one side about the right angle is double the other; if circles are described on these sides as diameters, prove that their common chord is two-fifths of the hypotenuse.

In the rt.- $\angle^d$   $\triangle ABC$  let the  $\angle A$  be a rt.  $\angle$  and let  $AB = 2AC$ . Let  $AI$

be the common chord of two circles  $ADC$  and  $ADB$  described on  $AC$  and  $AB$  as diameters.

It is reqd. to prove that  $AD = \frac{2}{3}$  of the hypotenuse  $BC$ .

**Proof.** Suppose the circle described on  $AC$  as diameter cuts the hypotenuse at  $D$ .

$\therefore$  the  $\angle ADC$  being in a semi-circle = a rt.  $\angle$ .

$\therefore$  the  $\angle ADB$  being adjacent to it = a rt.  $\angle$ .

$\therefore$  the circle described on  $AB$  as diameter passes through  $D$ . Now, in the rt.  $\angle^d \triangle ABC$ ,  $AD$  is perpendicular to the hypotenuse  $BC$ .

$\therefore BC \cdot BD = AB^2$  and  $BC \cdot CD = AC^2$ . (Cor., Theor. 9)

$$\frac{BC \cdot BD}{BC \cdot CD} = \frac{AB^2}{AC^2} = 4. \quad [\because AB = 2AC]$$

$$\text{i.e. } \frac{BD}{CD} = 4. \quad BD = 4CD.$$

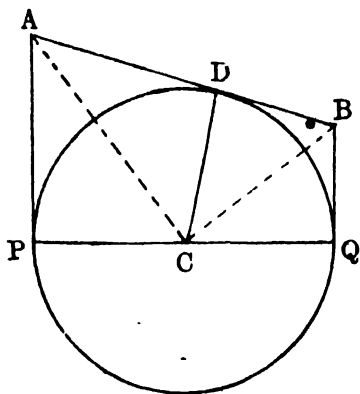
Now  $BC = BD + CD = 4CD + CD = 5CD$ .

$\therefore CD = \frac{1}{5}BC$ . Hence  $BD = \frac{4}{5}BC$ .

Again,  $AD^2 = BD \cdot CD = \frac{4}{5}BC \cdot \frac{1}{5}BC = \frac{4}{25}BC^2$ .

$\therefore AD = \frac{2}{5}BC$ .

**Ex. 6.** Any tangent to a circle is drawn to cut a pair of parallel tangents; prove that the radius of the circle is a mean proportional between the segments into which the tangent is divided at the point of contact.



Let  $PA$  and  $QB$  be two parallel tangents at the pts.  $P$  and  $Q$  on the circumference of a circle whose centre is  $O$  and let another tangent to the circle at the pt.  $D$  on it cut the two former tangents at  $A$  and  $B$ . Join  $CD$ .

It is reqd. to prove that  
 $CD^2 = AD \cdot DB$ .

**Proof.** Join  $CP$ ,  $CQ$ ,  $CA$ ,  $CB$ .

Since  $PA$ ,  $PB$  are tangents and  $CP$ ,  $CQ$  are radii, each of the  $\angle^s CPA$ ,  $CQB$  is a rt.  $\angle$  and  $PA$ ,  $QB$  are parallel.



$\therefore P, C, Q$  are collinear.

[ It can be easily proved by drawing through  $C$  a line  $\parallel$  to  $PA$  ].

Now in the  $\triangle APC, ADC$ , the  $\angle ADC = \text{the } \angle APC$  [ each being the  $\angle$  made by a radius with the tangent is a rt.- $\angle$  ],

the hypotenuse  $AC$  is common and  $CP = CD$  (being radii)

$\therefore$  the two triangles are equal in all respects.

$\therefore$  the  $\angle ACD = \text{the } \angle ACP$ , i.e. the  $\angle ACD = \frac{1}{2} \angle PCP$ .

Similarly, it can be proved that the  $\angle BCD = \frac{1}{2} \angle QCD$ .

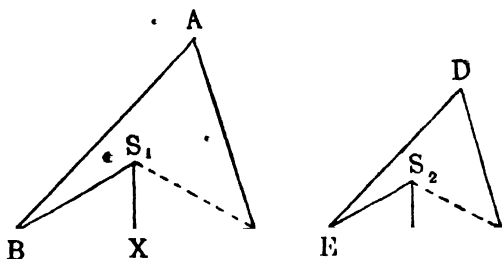
$\therefore$  the  $\angle ACD + \text{the } \angle BCD = \frac{1}{2}(\text{the } \angle PCP + \text{the } \angle QCD) = \text{a rt. } \angle$   
i.e. the  $\angle ACB = \text{a rt. } \angle$ .

Now, the  $\triangle ABC$  is a rt.- $\angle^d$   $\triangle$  and  $CD$  is perp. to the hypotenuse  $AB$ ,

[  $\because AB$  is a tangent and  $CD$  a radius ]

$\therefore CD^2 = AD \cdot DB$ .

**Ex. 7.** Prove that the areas of two similar triangles are to one another as the squares on the radii of their circum-circles.



Let the  $\triangle ABC, DEF$  be similar and  $S_1, S_2$  their circum-centres. Hence  $S_1B, S_2E$  are their circum-radii.

It is reqd. to prove that the  $\triangle ABC : \triangle DEF = S_1B^2 : S_2E^2$ .

**Construction.** From the pts.  $S_1, S_2$  draw the perps.  $S_1X, S_2Y$  to the sides  $BC, EF$  respectively. Join  $S_1C, S_2F$ .

**Proof.**  $S_1X, S_2Y$  are perps. to the chords  $BC, EF$  respectively from the pts.  $S_1, S_2$  the centres of the two circum-circles.

$\therefore BX = XC$  and  $EY = YF$ .

Now, in the  $\triangle BS_1X, CS_1X, S_1B = S_1C$  (being radii of the circum circle),  $BX = XC$  and  $S_1X$  is common.

$\therefore$  the two  $\triangle$ 's are equal in all respects.

$\therefore$  the  $\angle BS_1X = \text{the } \angle CS_1X = \frac{1}{2} \angle BS_1C$ .

Again,  $S_1$  is the centre of the circum-circle and  $A$  a pt. on its circumference,

$\therefore$  the  $\angle BS_1C = 2\angle BAC$ .  $\therefore$  the  $\angle BS_1X =$  the  $\angle BAC$ .

Similarly, the  $\angle ES_2Y =$  the  $\angle EDF$ .

But since, the  $\triangle ABC, DEF$  are similar, the  $\angle BAC =$  the  $\angle EDF$ .

$\therefore$  the  $\angle BS_1X =$  the  $\angle ES_2Y$ .

Hence, in the  $\triangle BS_1X, ES_2Y$ ,

the  $\angle BS_1X =$  the  $\angle ES_2Y$  (proved) and the  $\angle S_1XB =$  the  $\angle S_2YE$  (rt.  $\angle$ )

$\therefore$  the  $\triangle BS_1X, ES_2Y$  are equiangular, and hence similar.

$\therefore S_1B : S_2E = BX : EY = \frac{1}{2}BC : \frac{1}{2}EF = BC : EF$ .

Now, since the  $\triangle ABC, DEF$  are similar,

$$\begin{aligned}\triangle ABC : \triangle DEF &= BC^2 : EF^2 \\ &= S_1B^2 : S_2E^2.\end{aligned}$$

**Ex. 8.**  $ABCD$  is a cyclic quadrilateral of which the sides  $DA, CB$  are produced to meet in  $O$ . If  $AB$  is half of  $CD$ , prove that the quadrilateral  $ABCD$  is three times the triangle  $OAB$ .

Let  $ABCD$  be a cyclic quadrilateral in which  $CD = 2AB$  and let  $DA, CB$  be produced to meet in  $O$ .

It is reqd. to prove that the quadrilateral  $ABCD = 3\triangle OAB$ .

**Proof.** The exterior  $\angle OAB$  of the cyclic quadrilateral  $ABCD$  = the opposite interior  $\angle BCD$ .

Now, in the  $\triangle OAB, OCD$ ,  
the  $\angle OAB =$  the  $\angle BCD$  (proved) i.e. the  $\angle OCD$  and the  $\angle AOB =$  the  $\angle COD$  (the same angle).

$\therefore$  the  $\triangle OAB, OCD$  being equiangular are similar.

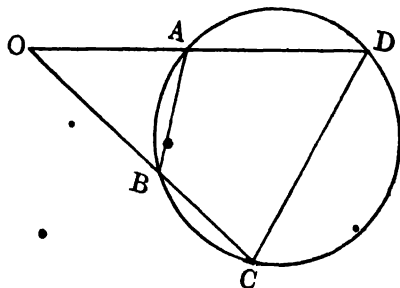
$$\therefore \triangle OCD : \triangle OAB = CD^2 : AB^2 = 4 : 1.$$

$$\therefore \triangle OCD = 4\triangle OAB.$$

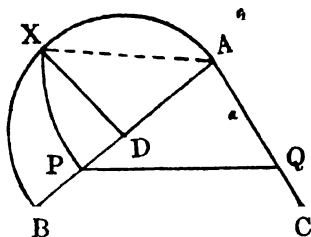
$$\begin{aligned}\text{Again the quadrilateral } ABCD &= \triangle OCD - \triangle OAB \\ &= 4\triangle OAB - \triangle OAB \\ &= 3\triangle OAB.\end{aligned}$$

**Ex. 9.** Show how to bisect a triangle by a straight line drawn parallel to the base.

Let  $ABC$  be the given  $\triangle$ .



It is reqd. to bisect the  $\triangle ABC$  by a straight line drawn parallel to the base  $BC$ .



**Construction.** Bisect  $AB$  at  $D$ . With centre  $D$  and radius  $AD$  describe a semi-circle on  $AB$ . At the pt.  $D$  in  $AB$  draw  $DX$  perp. to  $AB$ . Let  $DX$  cut the semi-circle at the pt.  $X$ . Join  $AX$ . From  $AB$  cut off  $AP = AX$ .

Through  $P$  draw  $PQ \parallel$  to  $BC$  to meet  $AC$  in  $Q$ .

Then  $PQ$  shall bisect the  $\triangle ABC$ .

**Proof.** Now,  $AP^2 = AX^2 = AD^2 + DX^2 = 2AD^2$  [  $\because AD = DX$ , being radii ]  
 $= \frac{1}{2}AB^2$ . [  $\because AB = 2AD$  ]  
 $\therefore AB^2 = 2AP^2$ .

Now, since  $PQ$  is  $\parallel$  to  $BC$ , the  $\triangle APQ$ ,  $ABC$  being equiangular are similar.

$$\therefore \triangle APQ : \triangle ABC = AP^2 : AB^2 = AP^2 : 2AP^2 = 1 : 2.$$

$$\therefore \text{the line } \triangle APQ = \frac{1}{2} \triangle ABC,$$

i.e. the line  $PQ$  bisects the  $\triangle ABC$ .

### Exercise II(C)

1. If the bisector of the vertical angle of a triangle also bisects the base, prove that the triangle is isosceles.

2. The angle  $BAC$  of a triangle  $ABC$  is bisected by  $AD$  meeting  $BC$  in  $D$  and the angle  $ABC$  is bisected by  $BI$  meeting  $AD$  in  $I$ ; prove that  $AI : ID = (AB + AC) : BC$ .

3. The angle  $BAC$  of a triangle  $ABC$  is bisected by  $AD$  meeting  $BC$  in  $D$  and  $O$  is the middle point of  $BC$ ; if  $AB > AC$ , prove that  $OB : OD = (AB + AC) : (AB - AC)$ .

4. From any point  $P$  on a circle described on  $AB$  as diameter,  $PC$  and  $PD$  are drawn making equal angles with  $AP$  and meeting  $AB$  in  $C$  and  $D$ ; prove that  $AD : DB = AC : CB$ .

5.  $AD$  is the median of a triangle  $ABC$  and the angles  $ADB$  and  $ADC$  are bisected by  $DE$  and  $DF$  meeting  $AB$  and  $AC$  at  $E$  and  $F$  respectively. Prove that  $EF$  is parallel to  $BC$ .

6. If the bisectors of one pair of opposite angles of a quadrilateral meet on one of the diagonals, prove that the bisectors of the other pair of opposite angles will meet on the other diagonal.

7. Prove that in any triangle the bisectors of two exterior angles and the bisector of the third interior angle are concurrent.

8. A straight line  $AB$  is divided internally at  $C$  and externally at  $D$  in the same ratio, and on  $CD$  as diameter a circle is described. If  $P$  be any point on this circle, show that  $PC$  and  $PD$  are the internal and external bisectors respectively of the angle  $APB$ .

9.  $AB$  is a fixed straight line and  $C$  a fixed point in it and through  $C$  any straight line  $CD$  is drawn; find a point  $P$  in  $CD$  such that  $AP : PB = AC : CB$ .

10. The internal and external bisectors of the vertical angle  $APB$  cut the base  $AB$  in  $C$  and  $D$  respectively; if  $O$  is the middle point of  $AB$ , prove that  $OB^2 = OC \cdot OD$ .

11.  $ABC$  is a triangle right-angled at  $A$  and  $AB$  is greater than  $AC$ ;  $AD$  is drawn perpendicular to  $BC$ . If  $BC$ ,  $AB$  and  $AC$  are in continued proportion, show that  $AC = BD$ .

12.  $AD$  and  $BE$  are the medians of the triangle  $ABC$  which meet at  $G$ ; if  $DE$  be joined, compare the areas of the triangles  $AGB$  and  $DGE$ .

13. Prove that the areas of two similar triangles are to one another as the squares on

(i) their corresponding altitudes;

(ii) their corresponding medians;

and (iii) the radii of their in-circles.

14.  $DEF$  is the pedal triangle of the triangle  $AEC$ . Prove that  $\triangle ABC : \triangle CDE = BC^2 : CE^2$ .

15. In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, show that the segments of the hypotenuse have the same ratio as the squares on the sides containing the right angle.

16.  $DE$  is drawn parallel to the base  $BC$  of a triangle  $ABC$  meeting  $AB$  and  $AC$  in  $D$  and  $E$  so that the triangle  $ADE$  is one-ninth of the whole triangle ; find the ratio of  $AD$  to  $DB$ .

17. If  $ABC$  be a triangle and  $D$  a point in  $BC$  such that  $BC$  is equal to the diagonal of the square on  $DC$ , show that a line through  $D$  parallel to  $AB$  bisects the triangle.

18.  $ABC$  is a triangle whose area is 16 sq. cm., and  $XY$  is drawn parallel to  $BC$ , dividing  $AB$  in the ratio 3 : 5 ; if  $BY$  is joined, find the area of the triangle  $BXY$ .

19. In a rectangle  $ABCD$ , square on  $AB$  is double that on  $AD$ . If perpendiculars be drawn from  $A$  and  $C$  on the diagonal  $BD$ , show that they will divide  $BD$  into three equal parts.

20. Show that the area of a regular hexagon described about a circle, is to the area of a regular hexagon inscribed in it as 4 to 3.

# Higher Secondary Elective Mathematics : Paper II

## •PLANE GEOMETRY

### ( Syllabus for Class X )

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle. [ *Theorem 11 Page 53* ]

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of one is equal to the rectangle contained by the parts of the other. [ *Theorem 12 Page 55* ]

[ *Note*—This proposition may be proved with the help of the properties of similar triangles. ]

### *Practical*

Construction of tangents to a circle and of common tangents to two circles (both cases). Construction of regular figures of 3, 4, 5 or 6 sides in or about a circle. [ *Problems 1, 2, 3, 4, 8, 9, 10, 11 Pages 67-82* ]

Construction of a mean proportional to two given straight lines. [ *Problem 15 Page 87* ]

Construction of a square equal in area to a given polygon. [ *Problem 13 Page 85* ]

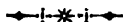
[ *Chapters III—IV, Pages 51 to 100* ]

# CONTENTS

| SUBJECTS   | PAGES |
|--|-------|
| <i>Chapter III : Secant and Tangent of a Circle</i>  |       |
| Secant and Tangent of a Circle   | 51    |
| Theorem 11   | 53    |
| Converse Theorem   | 54    |
| Theorem 12   | 55    |
| Converse Theorem   | 57    |
| Illustrative Examples  | 58    |
| Exercise III   | 65    |
| <i>Chapter IV : Practical Constructions</i>  |       |
| Practical Constructions  | 67    |
| Construction of Tangents (Problems 1, 2, 3 & 4)  | 67    |
| Construction of regular figures in or about a given circle (Problems 5, 6, 7, 8, 9, 10 & 11) | 71    |
| Construction of a Square equal in area to a given Polygon (Problems 12, 13 & 14)             | 84    |
| Construction of a Mean Proportional to two given straight lines (Problem 15)                 | 87    |
| Illustrative Examples  | 92    |
| Exercise IV  | 98    |

# PLANE GEOMETRY

( Course for Class X )



## CHAPTER III

### SECANT AND TANGENT TO A CIRCLE

#### 3.1. Secant and Tangent to a circle.

A straight line generally cuts a circle in two points. It is then called *secant* to the circle. When the two points of intersection of the secant with the circle coincide with one another, the secant is called a *tangent* to the circle. In Fig. 1 below the straight line  $PQ$  is a secant to the circle. In fig. 2 if the secant  $PQ$  be continuously turned towards the left about the point  $P$  in it, then the second point of intersection  $Q$  will gradually move towards, and finally coincide and become one with, the point  $P$ . Then in this limiting position the secant  $PQ$  becomes the tangent  $PT$ .

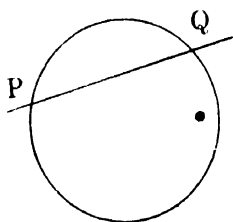


Fig. 1

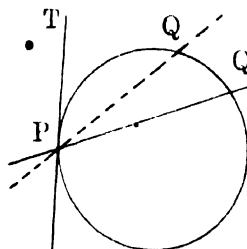


Fig. 2

Hence the straight line which touches a circle at one point only and does not meet it at any other point, even when produced, is called a *tangent* to the circle. The point on the circle at which the tangent touches it is called the *point of contact*.



### Contact of two circles.

Just as a straight line cuts a circle in two points, in a like manner two circles cut one another also in two points. If the two points of intersection of the straight line with the circle coincide with one another, the straight line touches the circle. In a similar way, if the two points of intersection of the two circles coincide with one another, the two circles are said to touch one another. The contact of two circles may be in two ways, internally or externally.

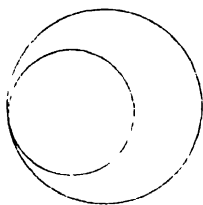


Fig. 1

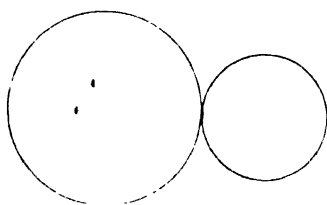


Fig. 2

When one circle lies entirely within the other and touches it (Fig. 1), the contact is said to be *Internal Contact*; and when one circle lies entirely outside the other and touches it (Fig. 2), the contact is said to be *External Contact*.

### Common Tangent to two circles.

When a straight line simultaneously touches two circles, then it is called a **Common Tangent** to the two circles. A common tangent may be of two kinds—direct and transverse. If the two points of contact of the common tangent to the two circles be on the same side of the line joining the centres of the two circles, the tangent is said to be a **Direct Common Tangent**, and if the two points of contact lie on both sides of the line joining the centres of the two circles, the tangent is said to be a **Transverse Common Tangent**.

Of two circles when one lies entirely within the other without touching it (Fig. 1), they have no common tangent. If the two circles touch internally (Fig. 2), they have only one common tangent at the point of contact. If the two circles touch

externally (Fig. 3), they have three common tangents of which one is at the point of contact and the remaining two are direct

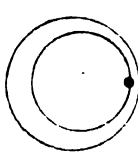


Fig. 1

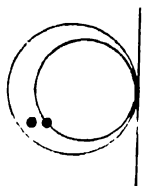


Fig. 2

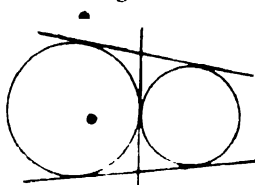


Fig. 3

common tangents. If the two circles cut one another (Fig. 4), they have two direct common tangents only and if without

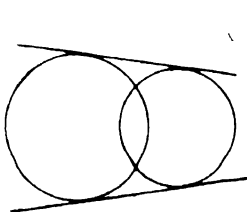


Fig. 4

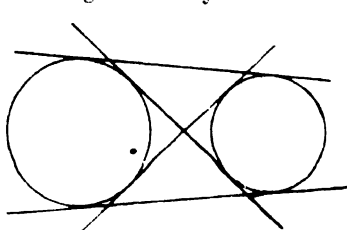


Fig. 5

cutting, one of the circles lies wholly outside the other, there are four common tangents—two direct and the other two transverse.

### Theorem 11

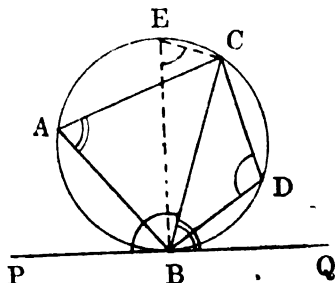
*The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.*

Let the straight line  $PQ$  touch the circle  $ABDC$  at the pt.  $B$  and  $BC$  be the chord of contact drawn from the pt.  $B$ .

It is reqd. to prove that

the  $\angle CBQ = \text{the } \angle BAC$  in the alternate segment,

and the  $\angle CBP = \text{the } \angle BDC$  in the alternate segment.



**Construction.** Draw the diameter through the pt.  $B$  and join  $EC$ .

**Proof.** Since the  $\angle ECB$  is a rt.  $\angle$  (being in a semi-circle)

$\therefore$  the  $\angle BEC +$  the  $\angle EBC =$  a rt.  $\angle$ .

Again,  $PBQ$  is a tangent and  $BE$  a diameter through the pt. of contact.

$\therefore$  the  $\angle EBQ =$  a rt.  $\angle =$  the  $\angle EBC +$  the  $\angle CBQ$ .

$\therefore$  the  $\angle BEC +$  the  $\angle EBC =$  the  $\angle EBC +$  the  $\angle CBQ$ .

Removing the common  $\angle EBC$  from both sides,

the  $\angle CBQ =$  the  $\angle BEC$ .

But the  $\angle BEC =$  the  $\angle BAC$  in the same segment.

$\therefore$  the  $\angle CBQ =$  the  $\angle BAC$ ; which is in the alternate segment.

Again, the  $\angle CBP$  is the supplement of the  $\angle CBQ$ .

$\therefore$  the  $\angle CBP$  is the supplement of the  $\angle BAC$ .

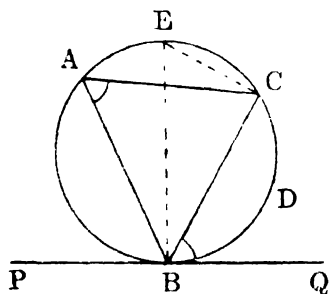
But  $ABDC$  is a cyclic quadrilateral.

$\therefore$  the  $\angle BAC$  is the supplement of the  $\angle BDC$ .

$\therefore$  the  $\angle CBP =$  the  $\angle BDC$ , which is in the alternate segment.

### Converse Theorem

*If through an extremity of a chord of a circle a straight line is drawn making with the chord an angle equal to the angle in the alternate segment, then the straight line so drawn touches the circle.*



Let  $BC$  be a chord of the circle  $ABDC$  and let the  $\angle CBQ$  made by the line  $PBQ$  drawn through the extremity  $B$  of the chord with  $BC$  be equal to the  $\angle BAC$  in the alternate segment.

It is reqd. to prove that the str. line  $PBQ$  is a tangent to the circle.

**Construction.** Draw the diameter of the circle through  $B$  and join  $EC$ .

**Proof.** Now the  $\angle BEC$  — the  $\angle BAC$  in the same segment  
 = the  $\angle CBQ$  (Hyp.).

Again, the  $\angle ECB$  = a rt.  $\angle$ , being in a semi-circle.

$\therefore$  the  $\angle EBC$  + the  $\angle BEC$  = a rt.  $\angle$ .

$\therefore$  the  $\angle EBC$  + the  $\angle CBQ$  = a rt.  $\angle$

[  $\because$  the  $\angle BEC$  = the  $\angle CBQ$ , proved ]  
*i.e.* the  $\angle EBQ$  = a rt.  $\angle$ .

But the line  $BE$  is a diameter,

$\therefore$   $PBQ$  is a tangent, because a tangent is at rt.  $\angle^s$  to the radius drawn through the pt. of contact.

### ✓ Theorem 12

*If two chords of a circle intersect, either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.*

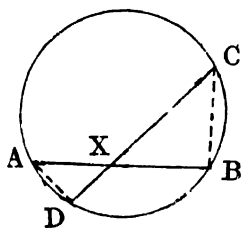


Fig. 1

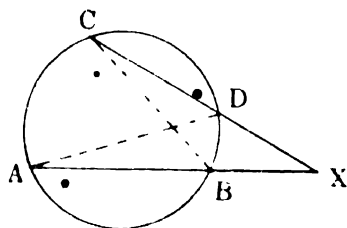


Fig. 2

Let the two chords  $AB$ ,  $CD$  of the circle  $ABC$  intersect, inside the circle in Fig. 1 and outside the circle in Fig. 2, at the point  $X$ .

It is reqd. to prove in both figures that  $AX \cdot XB = CX \cdot XD$ .

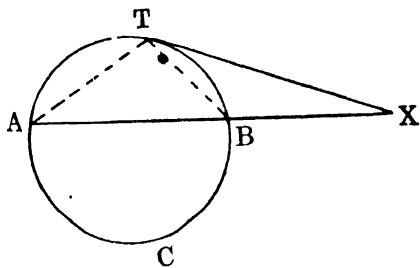
**Construction.** Join  $AD$ ,  $BC$ .

**Proof.** In the  $\triangle^s AXD$ ,  $BXC$ , the  $\angle AXD$  = the  $\angle BXC$   
 [ vertically opp.  $\angle^s$  in Fig. 1 and the same  $\angle$  in Fig. 2 ]  
 and the  $\angle XAD$  = the  $\angle XCB$  in the same segment.



**Cor.** If a chord of a circle passes through a point outside the circle, the rectangle contained by its parts is equal to the square on the tangent to the circle from the point.

Let the chord  $AB$  of the circle  $ACB$  be produced to the pt.  $X$  outside the circle and  $XT$  a tangent to the circle.



It is reqd. to prove that  $AX \cdot XB = XT^2$ .

**Construction.** Join  $AT$  and  $BT$ .

**Proof.** In the  $\triangle^s ATX$ ,  $BTX$ ,

the  $\angle BTX = \angle BAT$ , i.e. the  $\angle XAT$  in the alternate segment. [ $\because TX$  is a tangent]

and the  $\angle BXT = \angle AXT$  [the same angle]

$\therefore$  the  $\triangle^s ATX$ ,  $BTX$  being equiangular are similar.

$\therefore AX : XT = XT : XB$ .

$\therefore$  by cross-multiplication,  $XT^2 = AX \cdot XB$ .

**N. B.** An independent proof of this corollary is given here as it is of great importance. In fact it can be easily proved from Theorem 12. If the two points of intersection  $C$ ,  $D$  of the secant  $CDX$  gradually approach one another and ultimately coincide with the pt.  $T$ , the secant  $CDX$  becomes the tangent  $XT$  and  $CX = DX = XT$ . Hence  $CX \cdot XD = XT^2$ .

### Converse Theorem

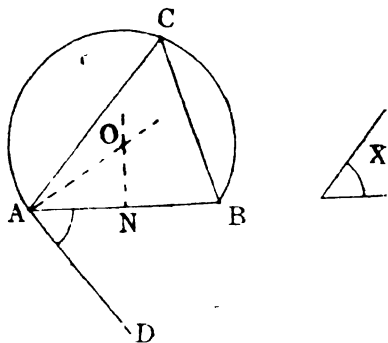
If two straight lines intersect internally or externally, so that the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other, then the extremities of the lines are concyclic.

This Converse Theorem can be easily proved, the proof whereof is left to the students as an exercise.

The converse of the above corollary is also true. Its proof is also left to students.

### 3.2. Illustrative Examples.

**Ex. 1.** On a given finite straight line show how to describe a segment of a circle which shall contain an angle equal to a given angle.



Let  $AB$  be the given straight line and  $X$  the given angle.

It is reqd. to describe on  $AB$  a segment of a circle which shall contain an angle equal to the given  $\angle X$ .

**Construction.** At the pt.  $A$  in  $AB$  make the  $\angle BAD$  equal to the  $\angle X$  and draw  $AO$  perp. to  $AB$ . From the middle pt.  $N$  of  $AB$  draw a line perp. to  $AB$  to meet  $AO$  at the pt.  $O$ . With centre  $O$  and radius equal to  $OA$  or  $OB$  describe an arc of a circle on the side of  $AB$  remote from  $D$ . Then this segment of the circle will be the required segment.

**Proof.**  $AD$  is a tangent to the circle at the pt.  $A$ , since it is at rt.  $\angle$  to the radius  $OA$ .

$\therefore$  the  $\angle BAD =$  the  $\angle ACB$  in the alternate segment.

i.e. the  $\angle ACB =$  the given  $\angle X$ .

**Ex. 2.** From an external point  $P$  tangents are drawn to a circle of centre  $O$  and  $OP$  cuts the chord of contact at  $Q$ ; show that (i)  $OP \cdot PQ = (\text{tangents})^2$  and (ii)  $OP \cdot OQ = (\text{radius})^2$ .

Let  $PA$  and  $PB$  be two tangents drawn from an external point  $P$  to the circle  $ABC$  whose centre is  $O$  and let  $AB$  be the chord of contact. Join  $OP$  and let it cut  $AB$  in  $Q$ .

It is reqd. to prove that (i)  $OP \cdot PQ = PA^2$

and (ii)  $OP \cdot OQ = OA^2$ .

**Construction.** Join  $OA$ ,  $OB$ .

**Proof.** Now in the  $\triangle OAP, OBP$ ,

$OA = OB$  (being radii),  $PA = PB$  [tangents to the circle from  $P$ ] and  $OP$  is common to both.

$\therefore$  the  $\triangle OAP, OBP$  are equal in all respects.

$\therefore$  the  $\angle OPA = \angle OPB$ .

Again, in the  $\triangle PAQ, PBQ$ ,

$PA = PB$ .  $QP$  is common to both and the included  $\angle QPA = \angle QPB$ .

$\therefore AQ = QB$  and the  $\angle AQP = \angle BQP = \text{a rt. } \angle$ . (adjacent  $\angle$ 's)

Since,  $OA, OB$  are radii and  $PA, PB$  tangents,

$\therefore$  each of the  $\angle OAP, OBP$  is a rt.  $\angle$ ,

$\therefore$  the four points  $O, A, P, B$  are concyclic.

$\therefore OQ \cdot QP = AQ \cdot QB = AQ^2$ .

Now (i)  $OP \cdot PQ = (OQ + QP) \cdot PQ = OQ \cdot PQ + PQ^2$   
 $= AQ^2 + PQ^2 = PA^2$  [  $\angle AQP = \text{a rt. } \angle$  ]

(ii)  $OP \cdot OQ = (OQ + QP) \cdot OQ = OQ^2 + OQ \cdot QP$   
 $= OQ^2 + AQ^2 = OA^2$ . [  $\angle AQP = \text{a rt. } \angle$  ]

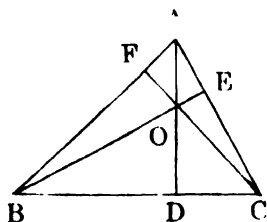
**Alternative Proof.**

In the  $\triangle OAP$ , the  $\angle OAP = \text{a rt. } \angle$  and  $AQ$  is perp. to the hypotenuse  $OP$ .

$\therefore$  by Cor. 2 and Cor. 3 of Theorem 9

*Ans*  $OP \cdot OQ = OA^2$  and  $OP \cdot PQ = PA^2$ .

**Ex. 3.**  $AD, BE$  and  $CF$  are the perpendiculars drawn from the vertices of a triangle  $ABC$  to the opposite sides and  $O$  is the orthocentre; show that  $AO \cdot OD = BO \cdot OE = CO \cdot OF$  and that  $AD \cdot AO = AB \cdot AF = AC \cdot AE$ .

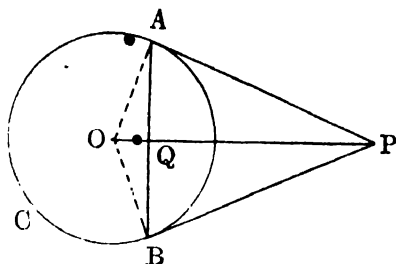


Let  $AD, BE, CF$  be perps. to the sides  $BC, CA, AB$  respectively of the  $\triangle ABC$  and  $O$  its orthocentre.

It is reqd. to prove that

(i)  $AO \cdot OD = BO \cdot OE = CO \cdot OF$ ,

and (ii)  $AD \cdot AO = AB \cdot AF = AC \cdot AE$ .





**Proof.** Since each of the  $\angle^s ADB, AEB$  is a rt.  $\angle$ ,

$\therefore$  the four points,  $A, B, D, E$  are concyclic and the chords  $AD, BE$  of this circle intersect at  $O$ .

$$\therefore AO \cdot OD = BO \cdot OE.$$

Similarly, it may be proved that  $BO \cdot OE = CO \cdot OF$ .

$$\therefore AO \cdot OD = BO \cdot OE = CO \cdot OF.$$

Again, each of the  $\angle^s BFO, BDO$  is a rt.  $\angle$ ,

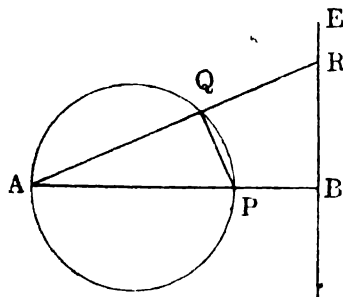
$\therefore$  the four points  $B, D, O, F$  are concyclic and the two chords  $OD, BF$  of this circle intersect at the pt.  $A$  outside the circle.

$$\therefore AB \cdot AF = AD \cdot AO.$$

Similarly, it may be proved that  $AD \cdot AC = AC \cdot AE$ .

$$\therefore AB \cdot AF = AD \cdot AO = AC \cdot AE.$$

**Ex. 4.**  $P$  is any point in a fixed straight line  $AB$ , and  $AQ$  is any chord of the circle described on  $AP$  as diameter;  $R$  is a point in  $AQ$  produced such that  $AQ \cdot AR = AP \cdot AB$ . Find the locus of  $R$ .



Let  $AB$  be the fixed straight line,  
 $P$  any pt. on it and  $AQ$  a chord of the circle described on  $AP$  as diameter.  
 Let  $R$  be a pt. in  $AQ$ , produced such that  $AQ \cdot AR = AP \cdot AB$ .

It is reqd. to find the locus of  $R$ .

**Proof.** Join  $BR$ .

$$\text{Now, } AP \cdot AB = AQ \cdot AR$$

$\therefore$  the four points  $P, Q, R, B$  are concyclic [Theorem 12, Converse]

$$\therefore \angle PQR + \angle PBR = 2 \text{ rt. } \angle^s.$$

But the  $\angle AQP$  is a rt.  $\angle$ , being in a semi-circle.

$\therefore$  the adjacent  $\angle PQR$  is also a rt.  $\angle$ .

$\therefore$  the  $\angle PBR$  is a rt.  $\angle$ .

$\therefore$  for all positions of the pt.  $R$ , it must be on the perp. to  $AB$  through  $B$ .

$\therefore$  this str. line must be the locus of  $R$ .

**Ex. 5.** Circles are drawn through two fixed points  $A$  and  $B$ , and from a fixed point in  $AB$  produced tangents are drawn to them; prove that the locus of the points of contact of these circles is a circle.

Let  $A, B$ , be two fixed points and  $P$  any point in  $AB$  produced; let  $PT$  be a tangent to any circle passing through  $A$  and  $B$ .

It is reqd. to show that the locus of  $T$  is a circle.

**Proof.** Now  $AP \cdot PB = PT^2$   
[ Cor. Theorem 12 ]

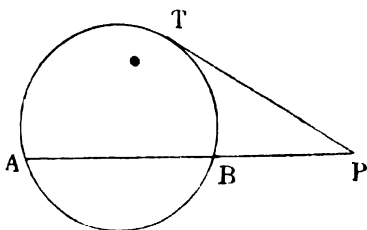
Again, since the three pts.  $A, B, P$  are fixed,

$\therefore AP \cdot PB = \text{a constant.}$

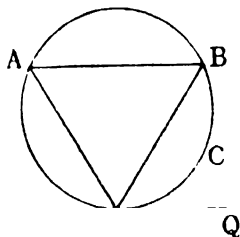
$\therefore PT^2 = \text{a constant i.e. } PT \text{ is constant.}$

Hence, the length of the tangent drawn to any circle passing through  $A$  and  $B$  from the fixed pt.  $P$  is constant.

$\therefore$  the locus of  $T$  is a circle with centre at  $P$ .



**Ex. 6.** If a tangent to a circle is parallel to a chord, the point of contact is the middle point of the arc cut off by the chord.



Let the tangent  $P'TQ$  at the pt.  $T$  of the circle  $ABC$  be parallel to the chord  $AB$ .

It is reqd. to prove that the pt. of contact  $T$  is the middle pt. of the arc  $ACB$  i.e. the arc  $AT =$  the arc  $BT$ .

**Construction.** Join  $AT, BT$ .

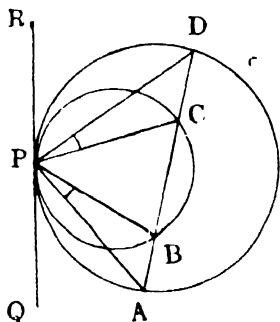
**Proof.** Since  $PT$  is  $\parallel$  to  $AB$ ,

the  $\angle ABT =$  the alternate  $\angle BTQ$

$=$  the  $\angle BAT$  in the alternate segment.

the arc  $AT =$  the arc  $BT$ .

**Ex. 7.** Two circles touch internally and a straight line is drawn to cut them; prove that the parts of it intercepted between the circles subtend equal angles at the point of contact.



Let two circles touch internally at the pt.  $P$  and let the line  $AD$  cut the two circles at the pt.  $A, B, C, D$  respectively.

It is reqd. to prove that the  $\angle APB$   
 $=$  the  $\angle CPD$ .

**Construction.** At  $P$ , the pt. of contact of the two circles, draw the common tangent  $QPR$ .

**Proof.** The  $\angle APB$   
 $=$  the  $\angle BPQ -$  the  $\angle APQ$   
 $=$  the  $\angle BCP -$  the  $\angle ADP$  i.e. the  $\angle CDP$   
 ( in the alternate segment )  
 $=$  the  $\angle CPD$ .

**Ex. 8.**  $AB$  is the diameter of a circle and  $PQ$  is a chord at right angles to it; prove that  $AP$  and  $BP$  are the bisectors of the angles formed by  $PQ$  with the tangent at  $P$ .

Let  $AB$  be the diameter of the circle  $APB$  and  $PQ$  a chord of the circle perp. to the diameter  $AB$ , let  $RT$  be the tangent to the circle at the pt.  $P$ .

It is reqd. to prove that  $AP$  and  $BP$  bisect the  $\angle RPQ$  and  $TPQ$  respectively.

**Construction.** Join  $AQ$  and let  $O$  be the pt. of intersection of  $AB$  and  $PQ$ .

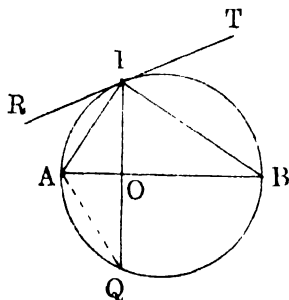
**Proof.** Since the diameter  $AB$  is perp. to  $PQ$ ,  $\therefore O$  is middle pt. of  $PQ$ .

$\therefore$  the arc  $AP =$  the arc  $AQ$ ,

$\therefore$  the  $\angle AQP =$  the  $\angle APQ$ .

Again, the  $\angle APR =$  the  $\angle AQP$  in the alternate segment.

$=$  the  $\angle APQ$ . ( proved )



$\therefore AP$  bisects the  $\angle RPQ$ .

Similarly, by joining  $BQ$  it may be proved that  $BP$  bisects the  $\angle TPQ$ .

**Ex. 9.**  $PT$  is a common tangent to two circles which intersect at  $A$  and  $B$ ; prove that the angles  $PAT$  and  $BAT$  are supplementary.

Let the two circles intersect each other at the pts.  $A, B$  and let  $PT$  be their common tangent.

It is reqd. to prove that the

$$\angle PAT + \text{the } \angle PBT = 2 \text{ rt. } \angle.$$

**Construction.** Join  $AP, AT, BP, BT$  and  $AB$ .

**Proof.** Since  $PT$  is a tangent to the circle and  $PA$  the chord of contact, the  $\angle APT = \angle ABP$  in the alternate segment.

Similarly, the  $\angle ATP = \text{the } \angle ABT$  in the alternate segment.

$$\therefore \text{the } \angle ABP + \text{the } \angle ABT = \text{the } \angle APT + \text{the } \angle ATP.$$

$$\text{i.e. the } \angle PBT = \text{the } \angle APT + \text{the } \angle ATP.$$

To each of these equals add the  $\angle PAT$ ,

$$\therefore \text{the } \angle PBT + \text{the } \angle PAT = \text{the } \angle APT + \text{the } \angle ATP + \text{the } \angle PAT = 2 \text{ rt. } \angle, \text{ the sum of the } \angle^s \text{ of a } \Delta.$$

**Ex. 10.** Two circles touch one another and from the point of contact straight lines are drawn to the extremities of a diameter of one of them; prove that these two straight lines meet the other circle at the extremities of a parallel diameter.

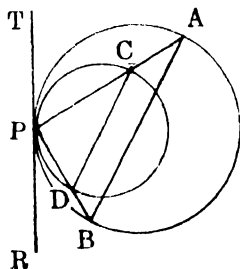


Fig. 1

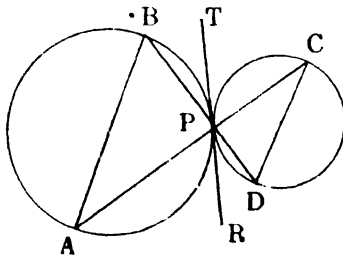


Fig. 2

Let two circles touch at the pt.  $P$  internally in Fig. 1 and externally in Fig. 2 and let  $AB$  be a diameter of one of them; let  $AP, BP$  be joined to

cut (in Fig. 1), and be produced to cut (in Fig. 2), the other circle at  $C, D$  respectively.

Join  $CD$ .

It is reqd. to prove that  $CD$  is a diameter of the second circle and is parallel to  $AB$ .

**Construction.** At  $P$  draw the common tangent  $TPR$  to the two circles.

**Proof.** Now, the  $\angle APB =$  a rt.  $\angle$ , being in semi-circle.

$=$  the  $\angle CPD$  [ the same  $\angle$  in Fig. 1 and

vertically opp.  $\angle$ ' in Fig. 2 ]

$\therefore CD$  is a diameter of the second circle.

Again, in Fig. 1, the  $\angle TPA =$  the  $\angle PBA$

[ in the alt. segment of the circle  $APB$  ]

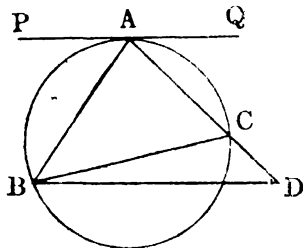
or, the  $\angle CDP$  [ in the alt. segment of the circle  $CPD$  ]

$\therefore$  the  $\angle CDP =$  the  $\angle PBA$ , but these are corresponding  $\angle$ '.

$\therefore AB$  and  $CD$  are parallel.

Similarly, it may be proved in Fig. 2 also that  $AB$  and  $CD$  are parallel.

**Ex. 11.**  $AB$  and  $AC$  are the chords of a circle  $ABC$  and  $BD$ , a straight line through  $B$  drawn parallel to the tangent at  $A$ , intersects  $AC$  produced in  $D$ . Prove that  $AB$  is a tangent to the circum-circle of the triangle  $BCD$ .



Let  $AB$  and  $AC$  be two chords of the circle  $ABC$ , the line  $PAQ$  be the tangent to the circle at the pt.  $A$  and the straight line  $BD$  drawn through  $B$  to the tangent  $PAQ$  intersect  $AC$  produced in  $D$ . Join  $BC$ .

It is reqd. to prove that  $AB$  is a tangent to the circum-circle of the  $\triangle BCD$ .

**Proof.** Now the  $\angle ADB =$  the alternate  $\angle DAQ$  [  $\because PQ$  and  $BD$  are  $\parallel$  ]

$=$  the  $\angle ABC$  in the alternate segment

[  $\because PQ$  is a tangent and  $AC$  the chord of contact ]

$\therefore$  the  $\angle ADB$  i.e. the  $\angle CDB =$  the  $\angle ABC$ .

$\therefore$  the line  $BA$  drawn from the extremity of the chord  $CB$  of the circum-circle of the  $\triangle BDC$  makes an angle with  $BC$  equal to the  $\angle BDC$  in the alternate segment of the circum-circle.

$\therefore AB$  is a tangent to the circum-circle of the  $\triangle BDC$ .

[ Converse of Theorem 11 ]

## Exercise III

1. If two circles intersect, prove that the tangents to the two circles from any point on their common chord produced, are equal.
2. Prove that the common chord of two intersecting circles, when produced, bisects their common tangent.

3. If two circles intersect, and two chords are drawn one in each circle through any point in their common chord, or that chord produced, prove that the extremities of these chords are concyclic.

4. If two circles touch one another, prove that any straight line drawn through the point of contact cuts off similar segments from the two circles.

[ Segments of circles containing equal angles are said to be *similar* ]

5. Two circles intersect at  $A$  and  $B$ , through any point  $P$  on the circumference of one of them, chords  $PA$  and  $PB$  are drawn to cut the other circle at  $C$  and  $D$ ; show that  $CD$  is parallel to the tangent at  $P$ .

6. If two circles touch one another, either internally or externally, and through  $A$ , the point of contact, two chords  $ABC$  and  $ADE$  (or  $BAC$  and  $DAE$ ) are drawn to cut the two circles at  $B, D$  and  $C, E$  respectively, prove that  $BD$  and  $CE$  are parallel.

7.  $AB$  is a common chord of two circles, one of which passes through  $O$ , the centre of the other; prove the  $OA$  bisects the angle between the common chord and the tangent to the first circle at  $A$ .

8. Two circles touch internally and a chord of the greater is a tangent to the smaller; prove that the chord is divided at its point of contact into segments subtending equal angles at the point of contact of the circles.

9. Prove that the perpendicular drawn from the middle point of an arc of a circle, to the chord of the arc, is equal to the perpendicular drawn from the same point to the tangent at either extremity of the chord.

10.  $AOB$  is a diameter of a circle whose centre is  $O$ , and  $C$  is the middle point of  $OA$ . Chord  $PCQ$  is drawn perpendicular to  $OA$ . Prove that the triangle  $PBQ$  is equilateral, and  $PQ^2 = 3OA^2$ .

11. Two tangents  $TA$  and  $TB$  are drawn to a circle whose centre is  $O$ . Another circle is drawn through  $T$  and touching  $AB$  at  $A$ , and  $OA$  is produced to meet this circle again at  $D$ . Prove that  $OA$  is equal to  $AD$ .

12. From a point  $P$ , tangents  $PA$ ,  $PC$  are drawn to a circle, and a secant  $PBD$  cuts the circle at  $B$  and  $D$ . Prove that in the quadrilateral  $ABCD$ , (i)  $AB \cdot CD = AD \cdot BC$ ,

$$(ii) AC \cdot BD = 2AB \cdot CD.$$

13. A chord  $AB$  of a circle is produced on both sides to  $P$  and  $Q$ , such that  $AP = BQ$ , and from  $P$  and  $Q$ , tangents  $PX$ ,  $QY$  are drawn to the circle on opposite sides of  $AB$ ,  $X$  and  $Y$  being the points of contact. Prove that the straight line  $XY$  bisects the chord  $AB$ .

14. Show how to construct a triangle having given the base, the vertical angle, and the ratio of the other two sides.

## CHAPTER IV

### PRACTICAL CONSTRUCTIONS

4.1. In the practical field construction of acute geometrical figures is of great importance. Of the different kinds of construction students have already become familiar with the easier ones like the *bisection of angles and straight lines, the drawing of a perpendicular to a given straight line at a given point in it or from a given point outside it, construction of an angle equal to a given angle at a given point in a given straight line, the drawing of a straight line through a given point parallel to a given straight line etc.*

In harder cases of construction, bisection of straight lines and angles, drawing of perpendiculars etc. are frequently resorted to. In these cases mere mention of what is to be done will do. How this is done need not be explained as the methods of these easy constructions are only too well known. Still the traces of these constructions should be left on the figure. An illustration will make the point clear. Suppose it is required to draw a tangent to a circle whose centre is  $O$  at a given point  $P$  on its circumference. Here it will be sufficient to say, "Draw  $PT$  perpendicular to  $OP$ ". It is not necessary to state with what radius and where the arc of a circle is to be described. But the traces of the construction of lines and arcs of circles necessary to draw the perpendicular must be left on the figure.

In Practical Geometry, the figures drawn should be neat and accurate as far as practicable. To make the figure neat and clean, lines should be drawn very carefully with the fine point of a hard pencil, and greatest care should be taken to see that there should be no occasion for erasing any line badly drawn. The traces of construction should be clear and well-defined.

#### 4.2. Construction of Tangents.

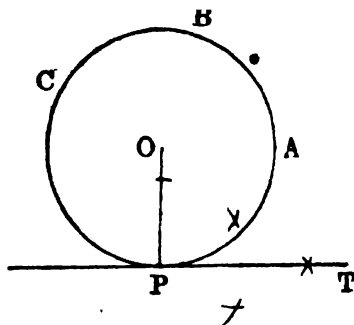
##### Problem 1

*Draw a tangent to a circle at a given point on the circumference.*

Let  $ABC$  be a circle whose centre is  $O$  and  $P$  the given point on the circumference.



It is required to draw a tangent to the circle  $ABC$  at the point  $P$ .



**Construction.** Join  $OP$ .

At  $P$  draw  $PT$  perp. to  $OP$ .

Then  $PT$  shall be tangent to the circle  $ABC$  at the point  $P$ .

**Proof.**  $O$  is the centre of the circle  $ABC$  and  $OP$  is its radius.

Now, the str. line  $PT$  is perp. to the radius  $OP$ .

$\therefore PT$  is tangent to the circle  $ABC$  at the pt.  $P$ .

## Problem 2

*Draw a tangent to a circle from a given external point.*

Let  $O$  be the centre of the circle and  $P$  a point outside it.

It is required to draw a tangent to the circle  $ABC$  from the point  $P$ .

**Construction.** Join  $OP$ . On  $OP$  as diameter describe a semi-circle intersecting the circle  $ABC$  at the point  $T$ . Join  $PT$ .

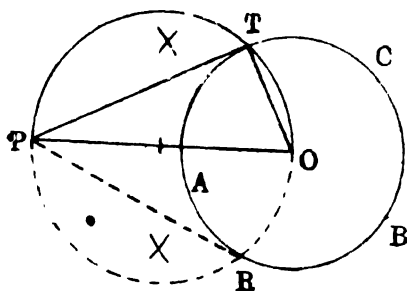
Then shall the str. line  $PT$  be tangent to the circle  $ABC$ .

**Proof.** Join  $OT$ .

Now, the  $\angle OTP$  being in a semi-circle is a rt.  $\angle$  and  $OT$  is a radius.

$\therefore$  the straight line  $PT$  being at rt.  $\angle^s$  to the radius  $OT$  is tangent to the circle  $ABC$ .

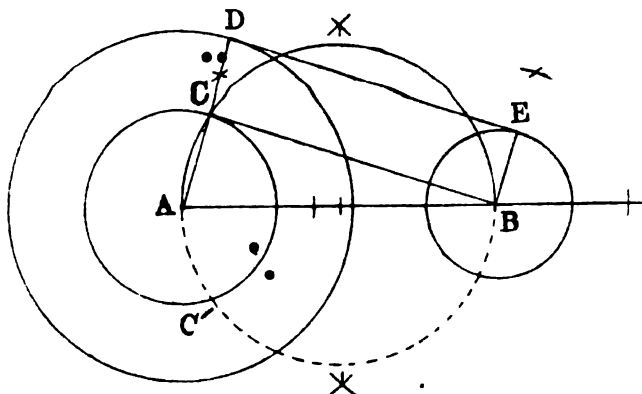
**N. B.** If the complete circle be drawn on  $OP$  as diameter, it will intersect the given circle at another point, say  $R$ . If  $PR$  be joined a second tangent to the circle  $ABC$  will be obtained. Hence two tangents can be drawn to a circle from a given external point.



## Problem 3

Draw a **direct** common tangent to two given circles.

Let  $A$  be the centre of the greater circle and  $a$  its radius and  $B$  be the centre of the smaller circle and  $b$  its radius.



It is reqd. to draw a common tangent to these two circles.

**Construction.** Join  $AB$ .

With centre  $A$  and radius  $(a - b)$ , the difference of the radii of the two circles, describe a circle. On  $AB$  as diameter another circle which cuts this circle at the point  $C$ . Join  $AC$  and produce it to meet the larger circle at  $D$ . Through  $B$  draw  $BE$  in the same direction as, and parallel to,  $AD$  to meet the smaller circle at  $E$ . Join  $DE$ .

Then shall  $DE$  be the direct common tangent to the two given circles.

**Proof.** Since  $AD = a$  and  $AC = a - b$ ,  $\therefore CD = b$ .

$\therefore CD$  and  $BE$  are equal and parallel.

$\therefore$  the figure  $BCDE$  is a parallelogram and the  $\angle CDE =$  the corresponding  $\angle ACB =$  a rt.  $\angle$ , being in a semi-circle.

$\therefore BCDE$  is a rectangle.

$\therefore$  each of the  $\angle^s BED, CDE$  is a rt.  $\angle$  and since  $AD, BE$  are the radii of the two given circles,  $\therefore DE$  is tangent to both the given circles.

Hence  $DE$  is a direct common tangent to the two given circles.

**Note 1.** The circle described on  $AB$  as diameter will intersect the circle with centre  $A$  and radius  $(a-b)$  at also another pt.  $C'$  besides the pt.  $C$ . Hence a second direct common tangent can be drawn on the other side of  $AB$ .

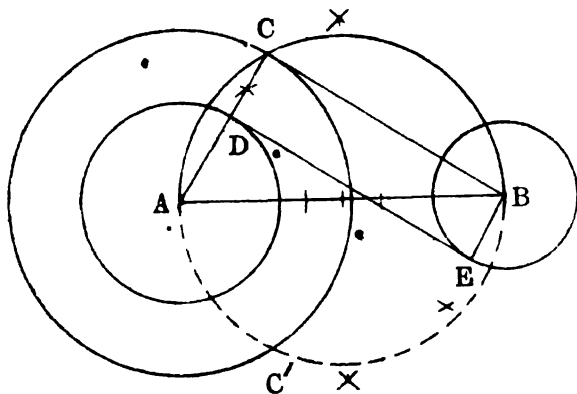
**Note 2.** With the same construction as given above direct common tangents to two given circles can be drawn even when they intersect one another.

**Note 3.** To draw direct common tangents to two equal circles, proceed as follows :—

Join the centres of the two given circles, and through their centres draw two radii on the same side of this joining line perpendicular to it. If the two extremities of the two radii so drawn on each side be joined, two direct common tangents to the two circles will be obtained.

#### Problem 4

*Draw a transverse common tangent to two given circles.*



Let  $A, B$  be the centres of the two given circles and  $a, b$  be their radii respectively.

It is reqd. to draw a transverse common tangent to the two given circles.

**Construction.** Join  $AB$ .

With centre  $A$  and radius equal to  $(a+b)$ , the sum of the two given circles, draw a third circle. From  $A$  draw a tangent  $BC$  to the third circle (i.e., on  $AB$  as diameter and describe a circle cutting the third circle at  $C$ . Join  $BC$ ). It of  $AC$  cutting the first circle at the pt.  $D$ . Now through  $B$  draw  $BE$  parallel and on the side opposite to  $AC$ . Join  $DE$ .

Then  $DE$  shall be the transverse common tangent to the two given circles.

**Proof.** Since  $AC = a+b$  and  $AD = a$ ,  $\therefore CD = b = BE$  and  $BE, CD$  are parallel (by construction).

$\therefore BE, CD$  are both equal and parallel.

Hence  $BEDC$  is a parallelogram.

Again,  $\angle C = a$  rt.  $\angle$ , being in a semi-circle.

$\therefore BEDC$  is a rectangle.

$\therefore$  the  $\angle ADE =$  the  $\angle BED = a$  rt.  $\angle$ .

Since  $AD, BE$  are radii,  $DE$  is a common tangent to both the circles.

**N. B.** Since the circle described on  $AB$  as diameter, intersects the circle with centre  $A$  and radius  $(a+b)$  at a second pt.  $C'$ , a second transverse common tangent to the two given circles can also be drawn.

If the two circles intersect one another, they have no transverse common tangent.

**4.3. Construction of regular figures in or about a given circle.**

A plane rectilinear figure whose sides are equal to one another and whose angles also are equal to one another is called a regular figure. From the definition of a regular figure an equilateral triangle and a square are regular figures. A regular pentagon is a five-sided figure all of whose sides are equal to one another and angles are also equal to one another.

The number of sides of a rectilinear figure is equal to the number of its angles i.e., if the number of sides be equal to 8, the number of its angles will also be 8. Again, if the number of sides of a rectilinear figure be known, the sum of its angles is also

known. Hence the magnitude of an angle of a regular rectilinear figure with a given number of sides may be easily determined. If the sum of the angles of a regular polygon with a given number of sides be divided by the number of sides of the polygon, the magnitude of an angle of the regular polygon can be found *e.g.* each angle of a regular pentagon is  $108^\circ$ , that of a regular hexagon is  $120^\circ$ , and so on.

Again, the number of angles subtended at the centre of a polygon by its sides is equal to the number of its sides. Whatever may be the number of sides of a polygon, the sum of the angles subtended by its sides at the centre is always equal to  $360^\circ$ . Hence, if  $360^\circ$  be divided by the number of sides of a regular polygon, it will give the magnitude of an angle subtended by a side of the regular polygon at its centre. If the magnitude of this angle be subtracted from  $180^\circ$ , it will give the magnitude of an angle of the regular polygon.

The centre of a regular polygon inscribed or circumscribed, in or about a circle is identical with the centre of the circle. Hence to inscribe or circumscribe a regular polygon in or about a circle, at first the angle subtended by a side of the polygon at its centre is determined by dividing  $360^\circ$  by the number of sides of the polygon and then an angle equal to this angle is drawn at the centre of the circle. The length of the straight line joining the two points at which the arms of the angle so drawn cut the circle gives the length of a side of the polygon. If chords equal in length to this side be successively placed inside the circle, the polygon will be inscribed in the circle as required. If tangents to the circle be drawn at the extremities of these chords, the circumscribed polygon will be obtained.

It is easy to inscribe or circumscribe an equilateral triangle, a square or a regular hexagon in or about a circle as the angles subtended by a side of these regular figures at the centre are  $120^\circ$ ,  $90^\circ$  and  $60^\circ$  respectively and these are easy to draw. But the angle subtended at the centre by the side of a regular pentagon is  $\frac{360^\circ}{5}$  or  $72^\circ$  and to draw this angle it is necessary to

know what medial section is. Hence before proceeding to inscribe or circumscribe regular figures in or about a circle, we shall define 'medial section' and discuss about it.

### Medial Section

**Definition.** A straight line is said to be divided in **medial section** when the rectangle contained by the whole line and one part is equal to the square on the other part; and the point of division is said to be the *point of medial section*. A straight line may be divided (i) internally or (ii) externally in medial section.



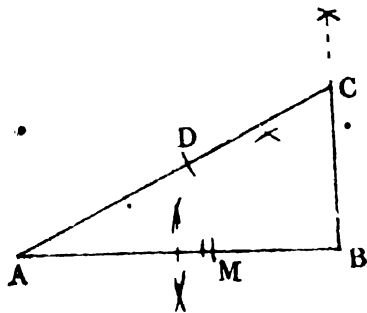
In the above figure (i), if  $AB \cdot BM = AM^2$ , then the line  $AB$  is said to be divided internally in medial section at the pt.  $M$ ; and (ii) if  $AB \cdot BM' = AM^2$ , then the line  $AB$  is said to be divided externally in medial section at the pt.  $M'$ .

### Problem 5

*Divide a given straight line internally in medial section.*

Let  $AB$  be the str. line to be divided in medial section.

**Construction.** At the point  $B$  in  $AB$  draw  $BC$  perp. to  $AB$ . Take  $BC = \frac{1}{2}AB$ . Join  $AC$ . From  $CA$  cut off  $CD = CB$ . Now, with centre  $A$  and radius equal to  $AD$  draw an arc cutting  $AB$  at the pt.  $M$ .



Then shall  $AB$  be divided internally at  $M$  in the medial section.

**Proof.** Let  $AB = a$  and  $AM = x$ .  $\therefore BM = a - x$ ,

$$BC = \frac{1}{2}a, AC = AD + DC = AM + BC = x + \frac{1}{2}a.$$

Now, since  $ABC$  is a rt.  $\angle^d \Delta$ ,

$$AC^2 = AB^2 + BC^2 \text{ i.e., } (x + \frac{1}{2}a)^2 = a^2 + (\frac{1}{2}a)^2.$$

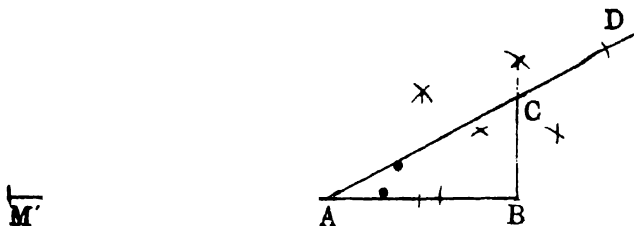
$$\therefore x^2 + ax = a^2, \quad \therefore x^2 = a^2 - ax = a(a - x)$$

$$\text{i.e., } AM^2 = AB \cdot BM.$$

$\therefore$  the str. line  $AB$  is divided internally at  $M$  in medial section.

### Problem 6

Divide a given straight line **externally** in medial section.



Let  $AB$  be the str. line to be divided externally in medial section.

**Construction.** At  $B$  in the str. line  $AB$  draw a perp. to  $AB$  and take  $BC = \frac{1}{2}AB$ . Join  $AC$  and produce it to  $D$  making  $CD = CB$ .

Now with centre  $A$  and radius equal to  $AD$  draw an arc of a circle which intersects  $BA$  produced at  $M'$ .

Then shall the str. line  $AB$  be divided externally in medial section at  $M'$ .

**Proof.** Let  $AB = a$  and  $AM' = x$ .

$$\therefore BM' = x + a, \quad AD = x, \quad CD = CB = \frac{1}{2}a.$$

$$AC = AD - CD = x - \frac{1}{2}a.$$

Now, since  $ABC$  is a rt  $\angle^d \Delta$ ,

$$AC^2 = AB^2 + BC^2 \quad \text{i.e., } (x - \frac{1}{2}a)^2 = a^2 + (\frac{1}{2}a)^2.$$

$$\therefore x^2 - ax = a^2. \quad \therefore x^2 = ax + a^2 = a(x + a)$$

$$\text{i.e., } AM'^2 = AB \cdot BM'.$$

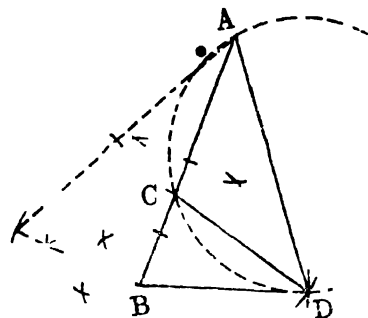
$\therefore$  the str. line  $AB$  is divided externally at  $M'$  in medial section

## Problem 7

Construct an isosceles triangle having each of the angles at the base double of the vertical angle.

It is reqd. to describe an isosceles triangle each of whose base angles is double of the vertical angle.

**Construction.** Take any str. line  $AB$ . Divide  $AB$  at  $C$  so that  $AB \cdot BC = AC^2$ , i.e., divide  $AB$  internally at  $C$  in media section.



With centres  $B$  and  $C$  and radius equal to  $AC$  draw two arcs of a circle and let them cut each other at  $D$ .

Join  $BD$ ,  $AD$ .

Then  $ABD$  shall be the reqd.  $\Delta$ .

**Proof.** Join  $CD$  and circumscribe a circle about the  $\Delta ACD$ .

Now  $AB \cdot BC = AC^2 = BD^2$ .

$\therefore BD$  touches the circle  $ACD$  at the pt.  $D$ .

$\therefore$  the  $\angle BDC =$  the  $\angle CAD$  in the alternate segment.

Again, by construction  $AC = CD$ ,  $\therefore$  the  $\angle CAD =$  the  $\angle CDA$ ,

$\therefore$  the  $\angle BDA =$  the  $\angle BDC +$  the  $\angle CDA =$  the  $\angle CAD +$  the  $\angle CDA = 2\angle CAD$  i.e.  $2\angle BAD$ .

Again, since  $BD = CD$ ,

$\therefore$  the  $\angle DBC =$  the  $\angle DCB =$  the  $\angle CAD +$  the  $\angle CDA$   
 $= 2\angle BAD$ ,  $\therefore AB = AD$ .

[  $\therefore \angle DBA$  i.e.  $\angle DBC = \angle BDA$ , each being  $= 2\angle BAD$  ]

$\therefore$  the  $\Delta ABD$  is isosceles and each of its base  $\angle^s$   $\angle ABD$ ,  $\angle ADB$  is double of the vertical  $\angle A$ .

**N. B.** Because  $\angle B = \angle D = 2\angle A$ ,

$\therefore \angle A + \angle B + \angle D = \angle A + 2\angle A + 2\angle A = 5\angle A$ .



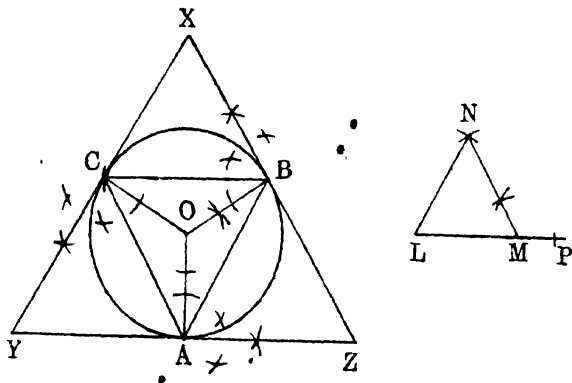
$\therefore 5\angle A = 180^\circ$  [ the sum of the  $\angle$ 's of the  $\triangle ABD$  ]

$\therefore A = 36^\circ$  and  $\angle B = \angle D = 72^\circ$ .

$\therefore$  now to draw an  $\angle = 72^\circ$ , if a triangle be described in the above manner the  $\angle ABD$  will be  $72^\circ$ .

### ✓ Problem 8

Construct a regular figure of three sides (that is, an equilateral angle) (i) in, or (ii) about a given circle.



Let  $O$  be the centre of the given circle.

It is reqd. to (i) inscribe an equilateral  $\triangle$  in the circle

and (ii) circumscribe an equilateral  $\triangle$  about the circle.

**(i) Construction.** Construct an equilateral  $\triangle LMN$ . Produce  $LM$  to  $P$ . Take any radius  $OA$  of the circle and at  $O$ , the centre of the circle, draw the  $\angle AOB =$  the  $\angle NMP$ , so that  $OB$  cuts the circle at  $B$ . Join  $AB$ . Place a chord  $BC$  equal to  $AB$  inside the circle. Join  $AC$ .

Then  $\triangle ABC$  shall be the reqd. equilateral  $\triangle$  inscribed in the circle.

**Proof.** The  $\triangle LMN$  is equilateral,  $\therefore$  each of its  $\angle$ 's  $= 60^\circ$ .

$\therefore$  the  $\angle NMP = 120^\circ$ .

$\therefore$  the  $\angle AOB$  drawn at  $O = 120^\circ$ .

Again, the chord  $BC =$  the chord  $AB$ .

$\therefore$  the  $\angle BOC =$  the  $\angle AOB = 120^\circ$ .

$\therefore$  the remaining  $\angle AOC = 120^\circ$

[  $\therefore$  sum of the  $\angle^s$  at  $O = 360^\circ$  ]

Now, since the three  $\angle^s$   $AOB$ ,  $AOC$  and  $BOC$  at the centre are equal to one another, the three chords,  $AB$ ,  $AC$ ,  $BC$  subtending these three angles are equal to one another.

$\therefore AB = AC = BC$ .

$\therefore$  the  $\Delta$  inscribed in the given circle is equilateral.

**(ii) Construction.** As before draw the  $\angle A$  and placing the chord  $BC$  equal to  $AB$  in the three pts.  $A, B, C$  on it. Draw three tangents  $A, B, C$  i.e., perp. to  $OA, OB, OC$ . Let these t one another at the pts.  $X, Y, Z$ .

Then  $\Delta XYZ$  shall be the reqd. equilateral about the circle.

**Proof.** Because  $ZY$  and  $ZX$  are tangent each of the  $\angle^s$   $OYZ, OZX =$  a rt.  $\angle$

[  $\therefore$  tangent and rad

$\therefore AOBZ$  is a cyclic quadrilateral

$\therefore$  the  $\angle AOB$  and the  $\angle AZB$  are supplem

But the  $\angle AOB = 120^\circ$  (by construction)

$\therefore$  the  $\angle AZB$  i.e. the  $\angle YZX = 60^\circ$ .

Similarly, it may be proved each of the  $\angle^s$   $XYZ, ZXY = 60^\circ$ .

$\therefore$  the  $\Delta XYZ$  is equilateral as the sides are tangents to the circle it is circumscribed about it.

### Problem 9

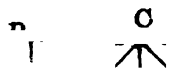
Construct a regular figure of four sides (that is a square) (i) in or (ii) about a given circle.

Let  $O$  be the centre of the given circle.

It is reqd. to (i) inscribe a square in the circle,

(ii) circumscribe a square about the circle.

**Construction.** Take a diameter  $AOB$  of the circle and at the centre  $O$  take another diameter  $COD$  at rt.  $\angle^s$  to  $AOB$  i.e. bisect  $AB$  at rt.  $\angle^s$ .



$\cap$

$\cap$

$D$

$N$

$B$

$M$

(i) Join  $AC, CB, BD$  and  $DA$ .

Then  $ACBD$  will be the reqd. regular quadrilateral i.e. square inscribed in the circle.

**Proof.** Because each of the  $\angle^s AOC, COB, BOD$  and  $DOA$  at the centre of the circle being a rt.  $\angle$ , the chords  $AC, CB, BD$  and  $DA$  are equal to one another.

Again each of the  $\angle^s ACB, CBD, BDA$  and  $\angle DAC$  being in a semi-circle, is a rt.  $\angle$ .

$\therefore ACBD$  is a regular quadrilateral, or square, inscribed in the circle.

(ii) At the pts.  $A$  and  $B$  draw two str. lines  $\parallel$  to  $CD$  and at the pts.  $C$  and  $D$  draw another two str. lines  $\parallel$  to  $AB$  and let them intersect at the pts.  $L, M, N, P$ .

Then  $LMNP$  shall be the reqd. regular quadrilateral or square circumscribed about the circle.

**Proof.**  $LM$  is  $\parallel$  to  $AB$  and  $AL$  and  $BM$  being each  $\parallel$  to  $CL$  are  $\parallel$  to one another.

$\therefore ALMB$  is a parallelogram,  $\therefore LM = AB$ .

Similarly,  $PN = AB$ .  $\therefore LM = PN = AB$ .

Similarly,  $PL = NM = CD$ . But  $AB = CD$

(diameters of the circle)

$\therefore LM = MN = NP = PL$ .

Again, by construction  $COBN$  is a parallelogram and the  $\angle COB =$  a rt.  $\angle$ .  $\therefore \angle N$  is a rt.  $\angle$ . Similarly the  $\angle^s L, M, P$  are also rt.  $\angle^s$ .

$\therefore$  the sides and angles of  $LMNP$  being equal to one another, the Fig.  $LMNP$  is a square.

Now the  $\angle COB$  of the parallelogram  $COBN$  being a rt.  $\angle$ , each of the  $\angle^s OCN$  and  $OBN$  is a rt.  $\angle$ .

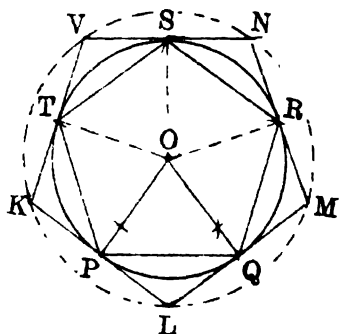
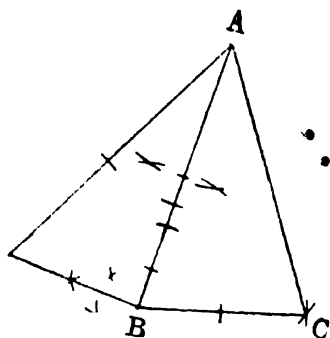
$\therefore$   $PN$  and  $MN$  touch the given circle at the pts.  $C, B$ .

Similarly,  $LM$  and  $LP$  are tangents to the circle.

$\therefore$   $LMNP$  is the reqd. square circumscribed about the circle.

### Problem 10

Construct a regular pentagon (i) in or (ii) about a given circle.



Let  $O$  be the centre of the given circle.

It is reqd. to (i) inscribe a regular pentagon in the circle,

and (ii) circumscribe a regular pentagon about the circle

**(i) Construction.** Describe an isosceles triangle  $ABC$  having each of the  $\angle^s ABC, ACB$  double of the vertical  $\angle BAC$ .

Take any radius  $OP$  of the circle and at the centre  $O$  of the circle draw the  $\angle POQ$  equal to the  $\angle ABC$ , so that the arm  $OQ$  cuts the circle at  $Q$ . Join  $PQ$ . Place three chords  $QR, RS, ST$  successively in the circle each equal to  $PQ$ . Join  $TP$ .

Then  $PQRST$  shall be the reqd. regular pentagon inscribed in the circle.

**Proof.** By construction, the  $\angle ABC = 72^\circ$  (Prob. 7)

$\therefore$  the  $\angle POQ = 72^\circ$

Since  $PQ, QR, RS, ST$  are equal chords of a circle,

$\therefore$  each of the  $\angle^s POQ, QOR, ROS, SOT$  at the centre of the circle  $= 72^\circ$ .

$\therefore$  the sum of these 4 angles  $= 4 \times 72^\circ = 288^\circ$ .

But the sum of the  $\angle^s$  at the pt.  $O = 360^\circ$ .

$\therefore$  the remaining  $\angle TOP = 360^\circ - 288^\circ = 72^\circ$ .

$\therefore$  the chord  $TP$  is equal to the other chords.

$\therefore$  the sides of the pentagon  $PQRST$  are equal to one another.

Again, the  $\triangle^s POQ, QOR$  etc. are equal in all respects and each of them is isosceles.

$\therefore$  the  $\angle OPQ = \angle OQP$ . Hence, each of the vertical  $\angle^s$  of the  $\triangle^s$  being  $72^\circ$ , their base angles are all equal to one another.

$\therefore$  the  $\angle PQR = \angle PQO + \angle OQR = 2\angle PQO$ .

$\therefore$  the  $\angle^s P, Q, R$  etc. of the pentagon being double the base-angle of these  $\triangle^s$  are equal to one another.

$\therefore$  the sides and angles of the pentagon  $PQRST$  being equal to one another, the pentagon is a regular one.

(ii) At the points  $P, Q, R, S, T$  of the circle, draw five perpendiculars  $KL, LM, MN, NV, VK$  to the radii  $OP, OQ, OR, OS, OT$  respectively.\* The pentagon  $KL MNV$  they have formed by meeting at the pts.  $K, L, M, N, V$  shall be the required regular pentagon circumscribed about the circle.

**Proof.** Since by construction  $KL, LM, MN, NV, VK$  are perps. to the radii at the pts.  $P, Q, R, S, T$  respectively on the circle, they are tangents to the circle.

Now, by construction, the  $\angle OPL + \angle OQL = 2 \text{ rt. } \angle^s$ .

$\therefore$  the four points  $P, O, Q, L$  are concyclic.

$\therefore$  the  $\angle L$  is the supplement of the  $\angle POQ$ .

Similarly, the  $\angle^s M, N$  etc. are the supplements of their respective opposite  $\angle^s QOR, ROS$  etc. at the pt.  $O$ .

But the five angles  $\angle^s POQ, ROS$  etc. at the pt.  $O$  are equal to one another.

---

\*The traces of construction of the perpendiculars have not been left on

$\therefore$  the five angles  $\angle L, \angle M, \angle N$  etc. being supplementary of these equal angles are equal to one another.

Again, each of the  $\Delta^s LPQ, QMR$  etc. is an isosceles  $\Delta$ .

[  $\because$  the two tangents drawn from each of the points  $L, M, N$  etc. to the circle are equal. ]

$\therefore$  the  $\angle LPQ = \angle LQP$ , the  $\angle MQR = \angle MRQ, \dots$

But the  $\angle^s L, M, N$  etc. of the  $\Delta^s LPQ, MQR, NRS$  etc. are equal to one another. (Proved)

$\therefore$  the  $\angle^s LPQ, LQP, MQR, MRQ$  etc are equal to one another.

Now, in the two  $\Delta^s LPQ$  and  $MQR$ , the  $\angle L = \angle M$ , the  $\angle LPQ = \angle MQR$  and  $PQ = QR$ .  $\therefore$  the two  $\Delta^s$  are equal in all respects.

$\therefore PL = MR$  and  $LQ = QM$ .

$\therefore LM = 2QM$ . Similarly,  $MN = 2MR$ .

But  $QM = LQ = LR = MR$ .  $\therefore LM = MN$ .

$\therefore$  the sides  $LM, MN$  etc. of the pentagon are all equal to one another.

$\therefore KLMNV$  is a regular pentagon and it has been circumscribed about the given circle.

### Alternative Method

Without drawing perpendiculars to the radii through the five pts.  $P, Q, R, S, T$ , the construction for circumscribing a regular pentagon about a circle may be shortened in the following manner.

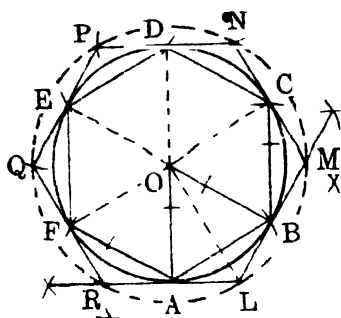
As before draw the  $\angle POQ = 72^\circ$  at the centre  $O$  of the circle. Draw perps. to the radii  $OP, OQ$  through the pts.  $P, Q$ . Let them intersect at  $L$ . With centre  $O$  and radius  $OL$  draw another circle. Let  $LP, LQ$  produced meet this circle at  $K, M$  respectively. Now, place the chords  $MN, NV$  equal to  $LM$  successively in this circle and join  $VK$ . Then shall  $LMNKV$  be the required regular pentagon about the given circle.

The proof is left to the students as an exercise.

## Problem 11

Construct a regular hexagon (i) or (ii) about a given circle.

Let  $O$  be the centre of the given circle.



It is reqd. to (i) inscribe or (ii) circumscribe a regular hexagon (i) in or (ii) about this circle.

**(i) Construction.** Take any radius  $OA$  of the circle and place successively five chords  $AB, BC, CD, DE, EF$  in the circle, each equal to  $OA$ .

Join  $AF$ .

Then shall  $ABCDEF$  be the regular hexagon inscribed in the given circle.

**Proof.** Join  $OA, OB, OC, OD, OE, OF$ .

Now the five  $\triangle AOB, BOC$  etc. each being an equilateral one, have each of their angles  $= 60^\circ$ .

Again, the sum of the  $\angle$ 's at  $O = 360^\circ$ .

But the sum of five  $\angle AOB, BOC$  etc.  $= 5 \times 60^\circ = 300^\circ$ .

$\therefore$  the remaining  $\angle AOF = 360^\circ - 300^\circ = 60^\circ =$  the  $\angle AOB$ .

$\therefore AF = AB = OA$ .

i.e. the  $\triangle AOF$  is also equilateral.

$\therefore$  the six sides of the hexagon are equal to one another and each of the six  $\angle$ 's  $A, B$  etc. being  $60^\circ + 60^\circ$  i.e.  $120^\circ$ , they are equal to one another.

Hence the hexagon  $ABCDEF$  is regular and it is inscribed in the given circle.

**(ii) Construction.** Take any radius  $OA$  of the given circle and place a chord  $AB$  in the circle equal to  $OA$ . Join  $OB$ .

At the points  $A$  and  $B$  draw perps. to  $OA$  and  $OB$  and let them intersect at  $L$ . Join  $OL$ .

Now, with centre  $O$  and radius  $OL$  draw another circle. Let  $LA, LB$  produced cut the circle at the pts.  $R$  and  $M$ . Beginning

with the pt.  $M$ , successively place three chords  $MN$ ,  $NP$ ,  $PQ$  each equal to  $LM$  in their circle. Join  $QR$ . Then shall  $LMNPQR$  be the reqd. regular hexagon circumscribed about the given circle.

**Proof.** By construction, the  $\triangle OAB$  is equilateral.

$$[\because OA = OB = AB]$$

$$\therefore \text{the } \angle AOB = 60^\circ.$$

Again, the rt.- $\angle$   $\triangle$ 's  $OAL$ ,  $OBL$  are equal in all respects

$$[\because \angle A \text{ and } B \text{ are rt. } \angle\text{'s}, OA = OB \text{ and the hypotenuse } OL \text{ is common}]$$

$$\therefore \text{the } \angle LOB = \text{the } \angle LOA = 30^\circ.$$

Now,  $LM$  is a chord of the outer circle and  $OB$  is perp. to it.

$$\therefore B \text{ is the middle point of } LM.$$

the  $\triangle$ 's  $OBL$ ,  $OBM$  are equal in all respects.

$$[\because BL = BM, OB \text{ is common and the rt. } \angle OBL = \text{rt. } \angle OBM]$$

$$\therefore \text{the } \angle MOB = \text{the } \angle LOB = 30^\circ, \therefore \text{the } \angle LOM = 60^\circ.$$

Similarly, the  $\angle LON = 60^\circ$ , and since the chords  $LM$ ,  $MN$ ,  $NP$ ,  $PQ$  of the outer circle are equal to one another, the  $\angle$ 's subtended by them at the centre are equal to one another.

$$\text{i.e., the } \angle MON = \text{the } \angle NOP = \angle POQ = \text{the } \angle LOM = 60^\circ \\ = \text{the } \angle LOB.$$

But the whole  $\angle$  round the pt.  $O = 360^\circ$ .

$$\therefore \text{the } \angle ROQ = 360^\circ - 5 \times 60^\circ = 60^\circ$$

$\therefore$  the chord  $RQ =$  the chord  $LM$  i.e. all the sides of the hexagon  $LMNPQR$  are equal.

Again, these sides being equal chords of the outer circle are equidistant from the centre  $O$  of the circle and this distance  $= OA =$  the radius of the given circle.

$\therefore$  the feet of the perps. [  $A, B$  and  $C, D, E, F$  say ] from the centre  $O$  upon the sides of the hexagon lie upon the given circle.

$\therefore$  the sides of the hexagon  $LMNPQR$  touch the given circle at the feet of the perpendiculars  $A, B, C, D, E, F$  i.e. the hexagon  $LMNPQR$  is circumscribed about the given circle.

Now  $OALB$  is a cyclic quadrilateral.

$$[\because \angle A \text{ and } \angle B \text{ are rt. } \angle\text{'s}]$$



$\therefore$  the  $\angle ALB +$  the  $\angle AOB = 180^\circ$ , but the  $\angle AOB = 60^\circ$ .

$\therefore$  the  $\angle ALB$  i.e. the  $\angle RLM = 120^\circ$ .

Similarly, each  $\angle$  of the hexagon  $LMNPQR = 120^\circ$ .

$\therefore$  the sides and angles of the hexagon being equal to one another, it is regular and is circumscribed about the given circle.

### Alternative method.

From any pt.  $A$  on the given circle, successively place five chords  $AB, BC, CD, DE, EF$  each equal to the radius of the circle.

Now, at the six pts.  $A, B, C, D, E, F$  draw six tangents (i.e. draw six perps. to the radii through these pts.) to the circle and let them intersect at the six pts.  $L, M, N, P, Q, R$ .

Then shall  $LMNPQR$  be the regular hexagon circumscribed about the circle.

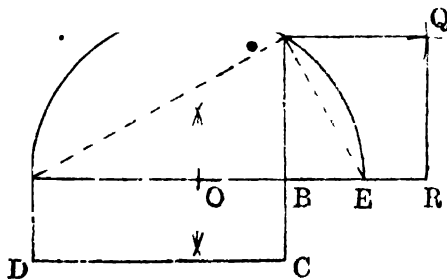
The proof is left to the students as an exercise.

N. B. In the First Method of drawing the circumscribed hexagon, the construction is comparatively short, although the proof is long. In Second Method, the proof is easier, but the construction is rather long.

## 4.4. Construction of a square equal in area to a given polygon.

### Problem 12

*Construct a square equal in area to the given rectangle.*



Let  $ABCD$  be the given rectangle.

It is reqd. to describe a square equal in area to the rectangle  $ABCD$ .

**Construction.** Produce the side  $AB$  of the rect.  $ABCD$  to  $E$  so that  $BE = BC$ .

Draw a semi-circle on  $AE$  as diameter. Produce  $CB$  to meet the semi-circle at  $P$ . Cut off  $BR$  from  $BE$  produced making it equal to  $BP$ . Now with centres  $P, R$  and radius equal to  $BP$  draw two arcs intersecting at the pt.  $Q$ . Join  $PQ, RQ$ .

Then shall  $PBRQ$  be the reqd. square.

**Proof.** It can be easily proved that  $PBRQ$  is a square.

Join  $AP, PE$ .

Now, the  $\angle APE = \text{a rt. } \angle$  (being in a semi-circle and  $PB$  is drawn from the rt.  $\angle$  perp. to the hypotenuse  $AE$ ),

$$\therefore PB^2 = AB \cdot BE = AB \cdot BC,$$

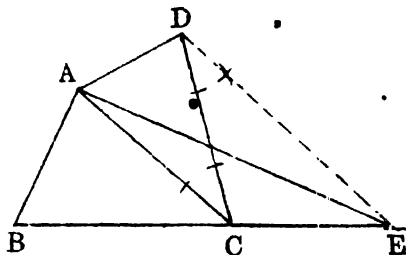
i.e. the square  $PBRQ = \text{the rect. } ABCD \text{ in area.}$

**N. B.** Construction of a square equal in area to a given triangle.

From the vertex of the given triangle draw a perp. to the base. The area of the rectangle with sides equal to this perpendicular and half the base is equal to the area of the given triangle. Hence, by the above Problem, a square drawn equal in area to this rectangle will be the square required.

### Problem 13

*Construct a triangle equal in area to a given quadrilateral.*



Let  $ABCD$  be the given quadrilateral.

It is reqd. to draw a triangle equal in area to the quadrilateral  $ABCD$ .

**Construction.** Join  $AC$ . Through  $D$  draw  $DE \parallel$  to  $AC$  to meet  $BC$  produced at  $E$ . Join  $AE$ .



## PRACTICAL CONSTRUCTIONS

If the work of construction be made with the same figure, it will render the figure untidy. So the construction at different stages is shown in three separate figures. In the first figure, a triangle equal to the pentagon, in the second, a rectangle equal to the triangle and in the third the square equal to the rectangle have been drawn.

Let  $ABCDE$  be the given pentagon.

It is reqd. to describe a square equal to the pentagon  $ABCDE$  in area.

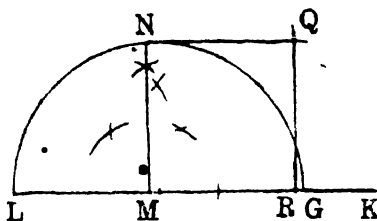
**Construction.** (i) Join  $AC$ ,  $AD$ . Throgh  $B$  draw  $BF \parallel$  to  $AC$  and through  $E$  draw  $EG \parallel$  to  $AD$  to meet  $DC$  and  $CD$  produced at  $F$  and  $G$  respectively. Join  $AF$ ,  $AG$ . The  $\triangle AFG$  is equal to the pentagon in area (Fig. 1).

(ii) Through  $A$  draw  $AX \perp$  to  $FG$ . Bisect  $FG$  at  $P$ . Through  $P$  draw  $PL$  perpendicular-bisector of  $FG$  to meet  $AX$  at  $L$ . From  $XL$  produced cut off  $LM = PG$ . Then  $LPGM$  is a rectangle and its area is equal to that of the  $\triangle AFG$  i.e. to that of the given pentagon (Fig. 2).

(iii) Take a str. line equal to  $LM$  and produce it to  $K$ . From  $MK$  cut off a length equal to  $MG$  of Fig. 2. On  $LK$  as diameter describe a semi-circle. At the pt.  $M$  draw  $MN$  perp. to  $LK$  cutting the semi-circle at  $N$ . (Fig. 3)

The square described on  $MN$  is the reqd. square,

The pentagon  $ABCDE$  = the  $\triangle AFG$  = the rect.  $LPGM$   
= the square  $MNQR$ .



x

Fig.

45. Construction of a mean proportional to two given straight lines.

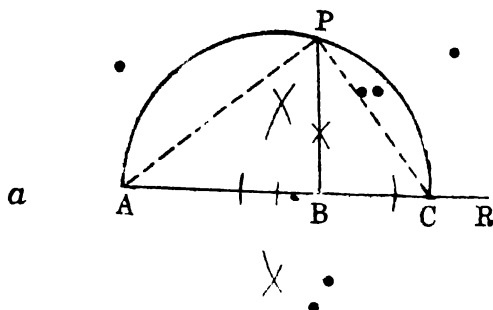
### Problem 15

Find a mean proportional to two given straight lines.

Let  $a$  and  $b$  be the two given str. lines.

It is reqd. to find the mean proportional between the str. lines  $a$  and  $b$ .

**Construction.** Take any str. line  $AR$  and from  $AR$  cut off  $AB$  and  $BC$  equal to  $a$  and  $b$  respectively. On  $AC$  as diameter describe a semi-circle. At the pt.  $B$  draw  $BP$  perp. to  $AC$  to meet the semi-circle at  $P$ .



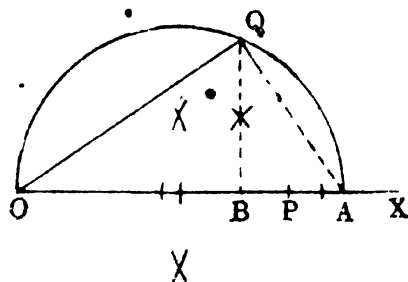
Then shall the str. line  $BP$  be the mean proportional between the str. lines  $AB$  and  $BC$  i.e. between the str. lines  $a$  and  $b$ .

**Proof.** Join  $AP$ ,  $PC$ . The  $\angle APC$  in the semi-circle = a rt.  $\angle$  and from the rt.  $\angle P$ ,  $PB$  is drawn perp. to the hypotenuse  $AC$ .

$$\therefore PB^2 = AB \cdot BC = a \cdot b$$

i.e.  $PB$  is the mean proportional between  $a$  and  $b$ .

**Alternative Method.**



**Construction.** From a str. line  $OX$  cut off  $OA$  and  $OB$  equal to  $a$  and  $b$ . Suppose  $OA > OB$ . On  $OA$  as diameter describe a semi-circle. At the pt.  $B$  draw  $BQ$  perp. to  $OA$  to meet the semi-circle at  $Q$ . Now from  $OA$  cut off  $OP$  equal to  $OQ$ .

## PRACTICAL CONSTRUCTIONS.

Then shall  $OP$  or  $OQ$  be the mean proportional between  $OA$  and  $OB$ .

**Proof.** Join  $OQ, QA$ .

Since the  $\angle OQA$  in the semi-circle = a rt.  $\angle$

= the  $\angle QBO$

(by construction),

and the  $\angle QOB$  is common,

$\therefore$  the  $\triangle^s AOQ$  and  $QOB$  are equiangular and hence similar.

$$\therefore \frac{OA}{OQ} = \frac{OQ}{OB} \quad OA \cdot OB = OQ^2 = OP^2.$$

$\therefore OP$  or  $OQ$  is the mean proportional between  $OA$  and  $OB$ .

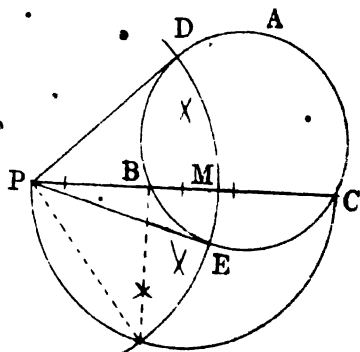
**N. B.** If it is required to find the mean proportional between two straight lines like  $OA$  and  $OB$  drawn from a pt. in the same direction, it is convenient to apply the Alternative Method and in the subsequent constructions this method has been resorted to.

**Corollary.** With the help of this problem we can draw a tangent to a given circle from an external point.

Let  $ABC$  be the given circle and  $P$ , a point outside it.

It is reqd. to draw a tangent to the circle  $ABC$  from  $P$ .

**Construction.** From the point  $P$  draw any secant  $PBC$  of the circle. Find the mean proportional  $PM$  between  $PB$  and  $PC$ . With centre  $P$  and radius equal to  $PM$  draw an arc of a circle to cut the given circle at the pts.  $D, E$ . Join  $PD, PE$ .



Then shall  $PD, PE$  be two tangents to the circle  $ABC$ .

**Proof.** Since  $PM$  is the mean proportional between  $PB$  and  $PC$ ,  $\therefore PB \cdot PC = PM^2 = PD^2$  or  $PE^2$ .

Hence  $PD$  and  $PE$  are tangents to the circle  $ABC$ ,

### 4.6. Illustrative Examples.

**Ex. 1.** Divide a given straight line in a given ratio (i) internally or (ii) externally.

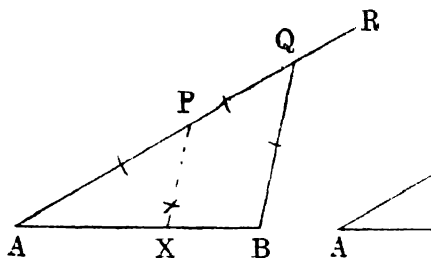


Fig. 1

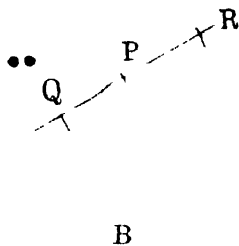


Fig. 2

Let  $AB$  be the given str. line and  $m : n$  the given ratio.

It is reqd. to divide  $AB$  (i) internally or (ii) externally in the ratio  $m : n$ .

**(i) Construction.** Draw any str. line  $AR$  through  $A$  and from  $A$  cut off two successive lengths  $AP, PQ$  so that  $AP = m$  and  $PQ = n$  (Fig. 1).

Now through  $P$  draw  $PX \parallel QB$  cutting  $AB$  at  $X$ .

Then shall  $AB$  be divided internally at  $X$  in the ratio  $m : n$ .

$$\text{i.e. } AX : XB = m : n.$$

**Proof.** Since  $PX$  and  $QB$  are parallel,

$$\therefore AX : XB = AP : PQ = m : n.$$

**(ii) Construction.** Draw any str. line through  $A$  and from it cut off  $AP$  equal to  $m$ . From  $P$  cut off  $PQ$  equal to  $n$ . (Fig. 2). Join  $BQ$ . Now through  $P$  draw  $PY \parallel QB$  to cut  $AB$  or  $BA$  produced at  $Y$ .

Then shall  $AB$  be divided externally at  $Y$  in the ratio  $m : n$ .

$$\text{i.e. } AY : YB = m : n.$$

**Proof.** Since  $PY$  and  $QB$  are parallel,

$$\therefore AY : YB = AP : PQ = m : n.$$

**Ex. 2.** Divide a given str. line  $AB$  internally at  $P$ , such that (i)  $AP^2 = 2PB^2$ , (ii)  $AP^2 = 3PB^2$ .

(i) Let  $AB$  be the given str. line.

It is reqd. to divide  $AB$  internally at  $P$  such that  $AP^2 = 2PB^2$ .

## PRACTICAL CONSTRUCTIONS

**Construction.** At  $A$  in  $AB$  draw the  $\angle BAC = 22\frac{1}{2}^\circ$  (At  $A$  draw a pt. to  $AB$  and bisect the rt.  $\angle$ ; then again bisect half the rt.  $\angle$ ). At  $B$  dr. perp. to  $AB$  and let it cut  $AC$  at  $C$ . Now from  $B$  cut off  $BP = BC$ .

Then shall  $AB$  be divided as reqd. at  $P$ .

i.e.  $AP^2$  will be equal to  $2BP^2$ .

**Proof.** Join  $PC$ .

Now, by construction,

$$BP = BC, \therefore \text{the } \angle BPC = \text{the } \angle BCP = 45^\circ \quad [\because \text{the } \angle ABC = 90^\circ]$$

But the  $\angle BPC = \text{the } \angle CPA + \text{the } \angle PCA$

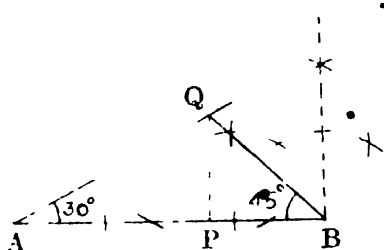
$$\text{i.e. } 45^\circ = 22\frac{1}{2}^\circ + \text{the } \angle PCA.$$

$\therefore$  the  $\angle PCA = 22\frac{1}{2}^\circ = \text{the } \angle PAC, \therefore AP = PC$ .

$$\begin{aligned} \therefore AP^2 &= PC^2 = PB^2 + BC^2 \quad [\because \text{the } \angle PBC = 90^\circ] \\ &= 2PB^2. \quad [\because PB = BC] \end{aligned}$$

(ii) Let  $AB$  be the given straight line.

It is reqd. to divide  $AB$  internally at  $P$  such that  $AP^2 = 3PB^2$ .



**Construction.** At  $A$  in  $AB$  draw the  $\angle BAQ = 30^\circ$  (first draw an  $\angle 60^\circ$  of an equilateral  $\Delta$  at  $A$  and then bisect it). At  $B$  make the  $\angle ABQ = 45^\circ$  (first draw a perp. to  $AB$  at  $B$  and then bisect the rt.  $\angle$ ). Let  $AQ$  and  $BQ$  meet at  $Q$ . From  $Q$  draw the perp.  $QP$  on  $AB$ .

Then shall  $AB$  be divided as reqd. at  $P$ .

i.e.  $AP^2$  will be equal to  $3PB^2$

**Proof.** The  $\angle QPB = 90^\circ$  and the  $\angle QBP = 45^\circ, \therefore$  the  $\angle PQB = 45^\circ$ .

$$\therefore QP = PB.$$

Again, the  $\angle QPA$  is a rt.  $\angle$  and the  $\angle QAP = 30^\circ$ .

$\therefore$  the hypotenuse = twice the smallest side  $QP$ .

$$\text{i.e. } AQ = 2QP.$$



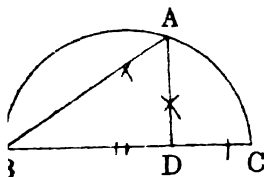
## PLANE GEOMETRY

Now, since the  $\angle APQ$  is a rt.  $\angle$ .

$$\begin{aligned}\therefore AP^2 &= AQ^2 - QP^2 = (2QP)^2 - QP^2 = 3QP^2 \\ &= 3PB^2, \quad [\because QP = PB, \text{ proved}] \end{aligned}$$

**Ex. 3.** Find geometrically the value of  $\sqrt{6}$ .

Take a str. line  $BC$ , 3 units in length and from it cut off  $BD = 2$  units long.



Find the mean proportional  $AB$  between  $BC$  and  $BD$ .

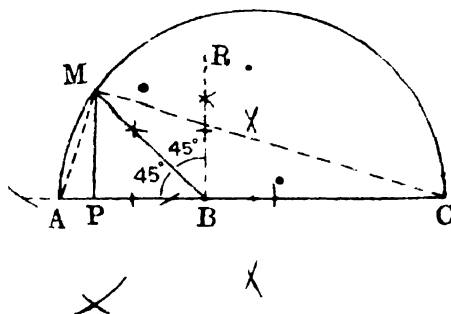
The str. line  $AB$  is the mean proportional between  $BC$  and  $BD$ .

$$\therefore AB^2 = BC \cdot BD = 3 \cdot 2 \text{ sq. units} = 6 \text{ sq. units}$$

$$\therefore AB = \sqrt{6} \text{ units.}$$

$\therefore$  the magnitude of the length of  $AB$  is the geometrical value of  $\sqrt{6}$ .

**Ex. 4.**  $A, B, C$  are three points in order in a straight line, and a point  $P$  on the straight line so that  $PB$  may be a mean proportional between  $PA$  and  $PC$ .



**Construction.** On  $AC$  as diameter describe a semi circle. At  $B$  draw perp. to  $AC$ . Bisect the rt.  $\angle ABB$  by  $BM$  meeting the semi-circle at  $M$ . In  $M$  draw  $MP$  perp. to  $AC$  meeting it at  $P$ .

Then  $P$  shall be the reqd. point.

**Proof.** Now, the  $\angle PBM = \angle PMB = 45^\circ$ .  $\therefore PB = PM$ .

Again, the  $\angle AMC$  in a semi-circle is a rt.  $\angle$ ,  $\therefore PM^2 = PA \cdot PC$ .

$$\therefore PB^2 = PA \cdot PC \quad [\because PM = PB]$$

Hence  $PB$  is a mean proportional between  $PA$  and  $PC$ .

## PRACTICAL CONSTRUCTIONS

**Ex. 5.** Having given the difference of the two adjacent sides of a rectangle construct it making its area equal to that of a given square.

Let the difference of the two adjacent sides of the reqd. rectangle be  $AB$  and  $N$ , a side of the given square.

It is reqd. to describe a rectangle whose area is equal to the square on the side  $N$  and the difference of whose adjacent sides is  $AB$ .

**Construction.** Take  $R$  the middle pt. of  $AB$  at  $R$  and draw  $RP$  perp. to  $AB$  making  $RP = N$ . Join  $BP$ . With centre  $B$  and radius  $BP$  describe semi-circle cutting  $BA$  or  $BA$  produced at  $C$ , and  $AB$  produced at  $D$ .

Then shall  $CR, RD$  be the reqd. rectangle.

**Proof.** Join  $CP, PD$ .

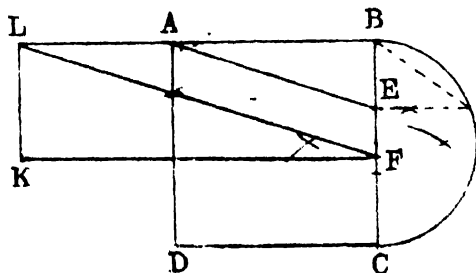
Now the  $\angle CPD$  in a semi circle = a rt.  $\angle$  and  $PR$  is perp. to the hypotenuse  $CD$ .

$\therefore CR, RD = PR^2 =$  the given square.

$$\begin{aligned} \text{Again, } RD - CR &= (BD + BR) - (BC - BR) \\ &= 2BR \quad [\because BP = BC] \\ &= AB. \end{aligned}$$

Hence the rect.  $CR, RD$  is the reqd. rectangle.

**Ex. 6.** Construct a rectangle with its sides in a given ratio and area equal to that of a given square.



Let  $ABCD$  be the given square and  $m:n$  the given ratio.

It is reqd. to describe a rectangle equal in area to the square  $ABCD$  and whose sides will be in the ratio  $m:n$ .

**Construction.** Take a pt.  $E$  in the side  $BC$  of the

## PLANE GEOMETRY

are so that  $AB:BE=m:n$  and find the mean proportional  $BF$  between  $AB$  and  $BE$ . Join  $AE$  and draw  $FL$  to  $EA$  to meet  $BA$  produced at  $L$ . Complete the rectangle  $FBLK$ .

Then shall  $BLKF$  be the reqd. rectangle.

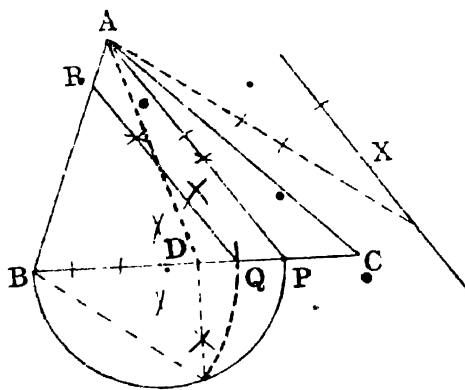
**Proof.** Square  $ABCD = BC^2$   
 Rectangle  $BLKF = BL \cdot BF$  [  $\because AB = BC$  ]  
 $= \frac{BE \cdot BC}{BF \cdot BF}$  [  $\because FL$  to  $AE$  ]  
 $= \frac{BC \cdot BE}{BF}$  [  $\because BF$  is the mean proportional between  $BC$  and  $BE$  ]

$\therefore$  the rect.  $BLKF =$  the square  $ABCD$ .

Again, because  $AE$  is to  $FL$ ,  $BL \cdot BF = AB \cdot BE = m \cdot n$ .

$\therefore$  the ratio of the sides  $BL$ ,  $BF$  is  $m:n$ .

**Ex. 7.** Bisect a triangle by a straight line drawn parallel to a given right line.



Let  $ABC$  be the given  $\triangle$  and  $X$  the given str. line.

It is reqd. to bisect the  $\triangle ABC$  by a straight line drawn parallel to  $X$ .

**Construction.** Bisect  $BC$  at  $D$  and through  $A$  draw  $AP$  to  $X$  to meet at  $P$ . Find  $BQ$  the mean proportional between  $BD$  and  $BP$ . Through  $Q$  draw  $QR$  to  $PA$  or  $X$  to meet  $AB$  at  $R$ .

Then shall the line  $QR$  bisect the  $\triangle ABC$ .

## PRACTICAL CONSTRUCTIONS

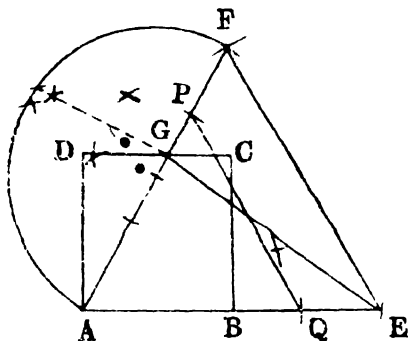
**Proof.** Join ...

Since the  $\triangle RBQ$  and  $\triangle ABP$  are similar,

$$\therefore \frac{\triangle RBQ}{\triangle ABP} = \frac{BQ^2}{BP^2} = \frac{BD \cdot BP}{BP^2} = \frac{BD}{BP} = \frac{\triangle ABD}{\triangle ABP}.$$

$\therefore$  the  $\triangle RBQ =$  the  $\triangle ABD = \frac{1}{2} \triangle ABC$ , [  $\because D$  is the middle pt. of  $BC$ . ]

**Ex. 8.** Construct an equilateral triangle equal in area to a given square.



Let  $ABCD$  be the given square.

It is reqd. to construct an equilateral triangle equal in area to the square  $ABCD$ .

**Construction.** Produce  $AB$  to  $E$  making  $BE = AB$ . Describe an equilateral  $\triangle AFE$  on the line  $AE$ . Let the side  $AF$  of this  $\triangle$  cut  $CD$  at the pt.  $G$ . Join  $GE$ . Find  $AP$  the mean proportional between  $AF$  and  $AG$ . Through  $P$  draw  $PQ$  to  $FE$  to meet  $FE$  at  $Q$ .

Then shall the  $\triangle APQ$  be the reqd. equilateral  $\triangle$ .

**Proof.** Since  $PQ$  is  $\perp$  to  $FE$ , the  $\triangle APQ$ ,  $\triangle AFE$  are similar.

$$\begin{aligned} \therefore \frac{\triangle AFE}{\triangle APQ} &= \frac{AF^2}{AP^2} = \frac{AF^2}{AG \cdot AF} = \frac{AF}{AG} \\ &= \frac{\triangle AFE}{\triangle AGE} \quad [\because E \text{ is the vertex of both } \triangle] \end{aligned}$$

$\therefore$  the  $\triangle APQ =$  the  $\triangle AGE$ .

But the  $\triangle AGE =$  the square  $ABCD$  [  $\because$  they stand between the same parallels and the  $\triangle AGE$  is on the base  $AE$  which is  $2AB$  ]

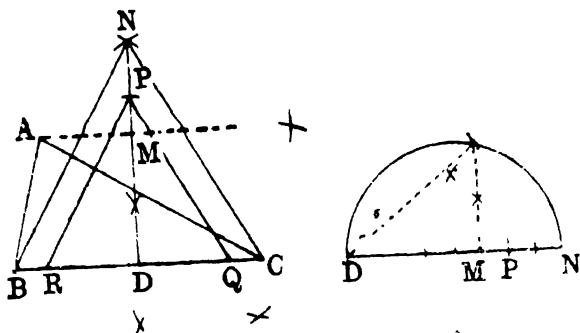
$\therefore$  the  $\triangle APQ =$  the sq.  $ABCD$ .

**Ex. 9.** Describe an equilateral triangle equal in area to a given triangle.

Let  $ABC$  be the given triangle.

It is reqd. to describe an equilateral triangle equal in area to the  $\triangle ABC$ .

**Construction.** Draw  $DMN$  the perp. bisector of  $BC$ .



Through  $A$  draw a str. line  $AM$  to  $BC$  and let it cut  $DMN$  at  $M$ . Then the  $\triangle MBC$  is isosceles and equal in area to the  $\triangle ABC$ .

With centre  $C$  and radius  $CB$  describe an arc of a circle to cut  $DMN$  at  $N$ . Then the  $\triangle NBC$  is an equilateral  $\triangle$  on the base  $BC$ .

Now find the mean proportional  $DP$  between  $DM$  and  $DN$ . Through the pt.  $P$  draw  $PQ \parallel$  to  $NC$  to cut  $DC$  at the pt.  $Q$ . From  $DB$  cut off  $DR = DQ$ . Join  $PR$ .

Then the  $\triangle PQR$  shall be the reqd. equilateral  $\triangle$ .

The proof is left to the students as an exercise.

**Ex. 10.**  $P$  is a given point within the angle  $\angle AOB$ . Draw a circle to pass through  $P$ , touching  $OA$  and  $OB$ .

Let  $P$  be the given pt. within the  $\angle AOB$ .

It is reqd. to draw a circle through  $P$  touching  $OA$  and  $OB$ .

**Construction.** Bisect the  $\angle AOB$  by  $OX$ . Draw  $PN$  perp. to  $OX$  and produce it to meet  $OA$  at the pt.  $R$ . From  $NR$  cut off  $NQ = NP$ .

Now, find the mean proportional  $RM$  between  $RP$  and  $RQ$ . On  $OA$  cut off from it  $RS$  and  $RT$  on both sides of  $R$ , making each of them equal to  $RM$ .

The circle passing through the three pts.  $S, P, Q$  will be the reqd. circle.

The circle passing through the three pts.  $T, P, Q$  will also be another such circle.



which touches  $OA$  and  $OD$ . Join  $OP$  cutting this circle at  $Q$ . Join  $QD$ . Through  $P$  draw  $PC \parallel$  to  $QD$  meeting  $OX$  at  $C$ . Now, the circle drawn with centre  $C$  and radius  $CP$  touches  $OA$  and  $OB$ .

This circle will be the reqd. circle.

**Proof.** From  $C$  draw  $CN$  perp. to  $OA$ .

Now, the  $\triangle ODQ, OCP$  are similar.  $[\because DQ \text{ is } \parallel \text{ to } CP]$

$\therefore DQ : CP = OD : OC = DM : CN$ .  $[\because DM \text{ is } \parallel \text{ to } CN]$

But  $DQ = DM$ ;  $\therefore CP = CN$ .

Hence, the circle with centre  $C$  and radius  $CP$  will pass through  $N$  and since the  $\angle CNA$  is a rt.  $\angle$ , this circle will touch  $OA$  at  $N$ .

As the pt.  $C$  is on the bisector of the  $\angle AOB$ , its perp. distances from  $OA$  and  $OB$  are equal.

$\therefore$  the circle will also touch  $OB$ .

**N. B.** The line  $OP$  cuts the circle with centre  $D$  at another pt.  $Q'$  besides  $Q$ . Hence by drawing a str. line through  $P$  parallel to  $Q'D$  another circle will be obtained.

### Exercise IV

1. Divide a right angle into five equal parts.
2. Find geometrically the values of (i)  $\sqrt{3}$ , (ii)  $\sqrt{5}$  and (iii)  $\sqrt{14}$ .
3. Circumscribe a circle about a regular hexagon.
4. Inscribe a square in a circle of 2 cm. radius and find its side by measurement and calculation.
5. Divide a given straight line into two parts, so that the rectangle contained by the two parts may be equal to a given square.
6. Having given the perimeter, construct a rectangle equal in area to a given (i) Rectangle, (ii) Square.
7. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.
8. Divide a given straight line  $AB$  internally at  $P$ , such that  $AP^2 = 5PB^2$ .

[Hints: Divide  $AB$  in medial section at  $M$ . From  $BA$  cut off  $BP'$  equal to  $\frac{1}{2}AM$ .]

9. Find a point on the circumference of a circle, from which lines drawn to two other given points on the circle will have given ratio.

10. Trisect a given triangle by lines drawn parallel to the base.

11. From the vertex  $A$  of a triangle  $ABC$  draw a straight line meeting  $BC$  produced in  $D$ , so that  $AD$  may be a mean proportional between the segments of the base.

[Hints: Make the  $\angle CAD = \text{the } \angle ABC$ .]

12. Draw a circle passing through two given points, and touching a given straight line, the given points being on the same side of the line, and the line joining the points being not parallel to the given straight line.

### Miscellaneous Exercise

1.  $ABC$  is a triangle obtuse-angled at  $A$ ;  $AD$  is drawn perpendicular to  $BC$ . Prove that

$$AB^2 + BC \cdot CD = AC^2 + BC \cdot BD.$$

2.  $ABC$  is a triangle; find a point  $P$ , within it such that  $PA^2 + PB^2 + PC^2$  may be a minimum.

3.  $ABC$  is an acute-angled triangle;  $AD, BE$  are perpendiculars to  $BC$  and  $CA$ , intersecting at  $O$ . Prove that

$$\text{the rect. } AD \cdot OD = \text{the rect. } BD \cdot DC.$$

4.  $ABC$  is an isosceles triangle whose vertex is  $A$ . If  $BP$  is perpendicular to  $AC$  meeting it at  $P$ , and  $PN$  perpendicular to  $BC$  meeting it at  $N$ , prove that

$$AB^2 = AN^2 + PN^2.$$

5. In a triangle  $ABC$ ,  $AD$  bisects  $\angle A$  and meets  $BC$  in  $D$ . The perpendicular to  $AD$  through  $D$  meets  $AB$  and  $AC$  in  $P$  and  $Q$ . Prove that  $AB : AC = BP : CQ$ .

6. If a straight line  $AB$  is divided internally in medial section at  $M$ , then  $AB^2 + BM^2 = 3AM^2$ .

7. Two circles intersect at  $A$  and  $B$ , and at  $A$  tangents are drawn, one to each circle to meet the circumferences at  $C$  and  $D$ . Show that  $AB$  is a mean proportional between  $BC$  and  $BD$ .



8. The common chord  $AB$  of two intersecting circles is produced to meet their common tangents at  $C$  and  $D$ ; if  $PQ$  be a common tangent, prove that

$$CD^2 = PQ^2 + AB^2.$$

9.  $AB$  is a chord of a circle, and from a point  $P$  on the circumference,  $PC$  and  $PD$  are drawn parallel to the tangents at  $A$  and  $B$  respectively, meeting  $AB$  at  $C$  and  $D$ . Prove that

$$AC : BD = PA^2 : PB^2.$$

[Hints:  $\triangle PAC, BPD$  are similar.]

10. Tangents  $PA, PB$  are drawn to a circle;  $BD$  is perpendicular to the diameter  $AC$  through  $A$ . Prove that  $PC$  bisects  $BD$ .

11. Two circles touch internally at  $A$ , and a chord  $BC$  of the bigger circle touches the smaller at  $T$ . If  $AB$  and  $AC$  cut the smaller circle at  $P$  and  $Q$  respectively, prove that

$$BT : CT = AP : AQ.$$

12.  $O$  is a fixed point and  $OP$  any line drawn to meet the circumference of a fixed circle in  $P$ ; if  $OP$  be divided at  $Q$  in a constant ratio, show that the locus of  $Q$  is a circle.

13. Construct an equilateral triangle, having given the sum of a side and altitude.

14. A regular hexagon has each side 2 cms. in length, Construct accurately to the scale a square equal to it in area.

15. Divide the hypotenuse of a right-angled triangle into two parts, such that the difference between the squares on them shall be equal to the square on one of the sides containing the right angle.

16. A square being given, construct a circle passing through one angular point, and touching two sides of the square.

[Hints: Divide the diagonal into two parts, such that the square on one part is twice that on the other.]

17. Describe a circle touching a given circle, and also touching a given straight line at a given point on it. How many such circles are possible?

18. Describe a circle touching a given circle, and passing through two given points outside the circle. How many such circles are possible?

# Higher Secondary Syllabus of Elective Mathematics :

## SOLID GEOMETRY (including Mensuration)

(Course for Class X)

### (a) Solid Geometry :

*Axiom* (i) One and only one plane may be made to pass through any two intersecting lines.

*Axiom* (ii) Two intersecting planes cut one another in a straight line and in no point outside it.

*To prove*

1. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

2. All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.

3. If two straight lines are parallel and if one of them is perpendicular to a plane, then the other is also perpendicular to the plane.

Concept of angle between two planes and angle between a straight line and a plane.

Concept of parallelism of planes.

Concept of a line being parallel to a plane.

Concept of skew lines.

### (b) Mensuration :

Parallelopipeds, Right Circular cones, Prisms and Pyramids (Expressions without proof, of the surfaces and volumes of the solids),

# CONTENTS

| CHAPTER   | PAGE |
|---|------|
| <b>1. Fundamental Concepts and Definitions</b>                      | 1    |
| <b>2. Axioms and Theorems</b>                                       | 6    |
| Theorem 1   | 6    |
| Theorem 2   | 10   |
| <b>3. Volumes and surface areas of regular Solids (Mensuration)</b> | 17   |
| Parallelopiped  | 17   |
| Rectangular parallelopiped  | 17   |
| Cube  | 19   |
| Prism   | 19   |
| Pyramid   | 20   |
| Tetrahedron   | 21   |
| Right Circular Cone   | 21   |
| Right Circular Cylinder   | 23   |
| Sphere  | 23   |

## IMPORTANT FORMULÆ AND RESULTS

### Solid Geometry (Mensuration)

1. *Rectangular paralleliped* (or cuboid).

If  $a, b, c$  be its length, breadth and height

(i) Area of the surface  $= 2(bc + ca + ab)$ .

(ii) Volume  $= abc$ .

(iii) Surface area of a cube of side  $a = 6a^2$ .

(iv) Volume  $= a^3$ .

2. *Right Pyramid on any regular base*

(i) Slant surface  $= \frac{1}{2}(\text{perimeter of base}) \times \text{slant height}$ .

(ii) Volume  $= \frac{1}{3}(\text{area of base}) \times \text{height}$ .

3. *Tetrahedron*.

$$\text{Volume} = \frac{1}{3}(\text{area of base}) \times \text{height}.$$

4. *Right Prism*.

(i) Lateral surface  $= (\text{perimeter of base}) \times \text{height}$ .

(ii) Volume  $= (\text{area of base}) \times \text{height}$ .

5. *Right circular cylinder*.

If  $r$  is the radius of the base and  $h$  the height of the cylinder,

(i) Area of the curved surface

$$= (\text{circumference of base}) \times \text{height}$$

$$= 2\pi rh.$$

(ii) Area of the whole surface

$$= 2\pi rh + 2\pi r^2 = 2\pi r(h + r).$$

(iii) Volume  $= (\text{area of base}) \times \text{height} = \pi r^2 h$ .

**6. Right circular cone.**

If  $r$  is the radius of the base,  $h$  the height,  $l$  the slant side and  $\alpha$  the semi-vertical angle of the cone,

(i) Area of curved surface

$$\begin{aligned} &= \frac{1}{2}(\text{circumference of base}) \times \text{slant side} \\ &= \frac{1}{2} \cdot 2\pi r \cdot l = \pi r l \\ &= \pi r \sqrt{h^2 + r^2} = \pi r^2 \operatorname{cosec} \alpha. \end{aligned}$$

(ii) Area of the whole surface  $= \pi(l + r)$ .

(iii) Volume  $= \frac{1}{3}(\text{area of base}) \times \text{height}$   
 $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \tan^2 \alpha.$

**7. Sphere.**

If  $r$  be the radius of the sphere,

(i) Area of curved surface  $= 4\pi r^2$ .

(ii) Volume  $= \frac{4}{3}\pi r^3$ .

# SOLID GEOMETRY

( To be taught in Class X )

## CHAPTER I

### FUNDAMENTAL CONCEPTS AND DEFINITIONS

**1.1.** We give below definitions and chief characteristics of some fundamental entities used in Solid Geometry.

(i) A **point** has position but no magnitude ; that is, it has neither length nor breadth nor thickness.

(ii) A **line** has length but no breadth and thickness.

(iii) A **surface** has length, breadth but no thickness.

(iv) A **solid** has length, breadth and thickness.

Thus, a brick is a *solid*, one of its six faces is a *surface*, an edge is a *line* and a corner is a *point*.

Each of the three elements (1) length, (2) breadth and (3) thickness of a body is called a *dimension* of the body.

Thus, a point has no (or zero) dimension, a line has one dimension, a surface has two dimensions and a solid has three dimensions.

A solid is bounded by surfaces, a surface is bounded by lines and a line is bounded by points.

**Solid Geometry** deals with the properties of lines, surfaces and solids in three dimensional space.

**1.2.** If a surface be such that the straight line joining *any two* points in it lies wholly on the surface, it is called a *plane surface* or more simply a *plane*.

**Note.** In this treatise straight lines are supposed to be of infinite length and the planes of infinite extent unless anything to the contrary is stated. The expression *lies wholly on the surface* means that every point in the straight line however produced both ways lies in the surface.

**1.3.** Lines are said to be **co-planar** if they be in a plane or a plane can be made to pass through them.

1'4. Two straight lines are said to be *parallel*, when they *lying in the same plane*, do not meet however far they may be produced both ways.

It should be noted that every pair of parallel straight lines is co-planar.

1'5. Lines are said to be *skew* when a plane cannot be made to pass through them and they do not meet however far they may be produced.

Thus, it should be noted that skew lines neither intersect nor are they parallel.

From (1'3), (1'4), (1'5), it is clear that two straight lines are either co-planar or skew.

If they are co-planar, they either intersect or are parallel and if they are not co-planar *i.e.*, if they are skew, they neither intersect nor are they parallel.

1'6. **Planes** are said to be **parallel** when they do not meet even if they are indefinitely produced in any direction.

1'7. A **straight line** and a **plane** are said to be **parallel** when they do not meet even if produced indefinitely in any direction.

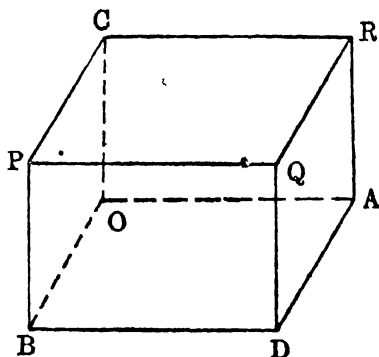


Fig. 1

In the above figure, the lines  $PQ$ ,  $BD$  are parallel; they lie in the plane  $PQDB$ .  $PQ$ ,  $OB$  are skew lines. Planes  $PQDB$ ,  $OARC$  are parallel. Straight line  $PQ$  is parallel to the plane  $OARC$ .

## FUNDAMENTAL CONCEPTS AND DEFINITIONS

1'8. A straight line is said to be **perpendicular** to a plane, if it is perpendicular to every straight line which meets it in that plane.

Such a straight line is said to be **normal** to that plane.

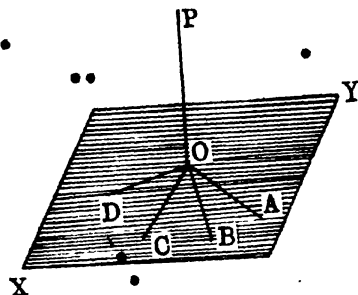


Fig. 2

Thus,  $PO$  is said to be perpendicular to the plane  $XY$  if it is perpendicular to every straight line  $OA$ ,  $OB$ ,  $OC$ ,  $OD$  etc. drawn in the plane to meet it at  $O$ .

1'9. If a straight line is parallel to the direction of a plumb-line hanging freely at rest, it is called a **vertical line**. The plane which is perpendicular to the vertical line is called the **horizontal plane**.

Any straight line drawn in the horizontal plane is called a **horizontal line**.

1'10. The *angle between two skew straight lines* (i.e., two non-co-planar straight lines) is measured by the angle contained by one of them and a straight line drawn through any point in that line parallel to the other.

Let  $AB$  and  $CD$  (Fig. 3) be two skew straight lines; through any point  $P$  on  $AB$  draw the straight line  $PQ$  parallel to  $CD$ . Then  $\angle QPB$  is the angle between the skew st. lines  $AB$  and  $CD$ .

Since, the sides of a triangle all lie in one plane, a triangle is a plane figure; a parallelogram is also a plane figure, but all the sides of a quadrilateral need not lie in one plane, so a quadrilateral may or may not be a plane figure. If a quadrilateral be drawn such that two of its adjacent sides lie in one plane and the other two in another plane, such a quadrilateral is called



## SOLID GEOMETRY

**skew or gauche.** If the extremities of a pair of finite skew lines be joined, a skew quadrilateral is formed.

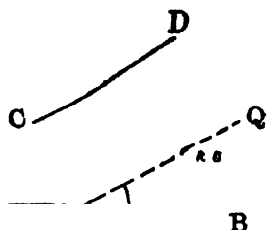


Fig. 3

That the sides of a quadrilateral need not lie in one plane can be easily seen by folding a plane quadrilateral about either diagonal.

1.11. The locus of the feet of the perpendiculars drawn from all points in a line on a given plane is called the *orthogonal projection* (or simply the *projection*) of the line on the plane.

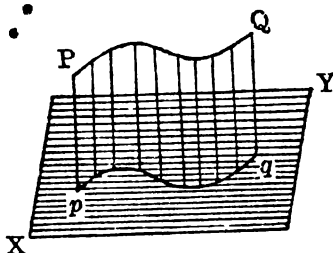


Fig. 4

In the above figure (4), the projection of the line  $PQ$  on the plane  $XY$  is the line  $pq$ .

*The projection of a straight line is itself a straight line.* A straight line and its projection are co-planar.

In the adjoining figure (5), the straight line  $pq$  is the projection of the straight line  $PQ$ .  $PQ$ ,  $pq$  are co-planar.

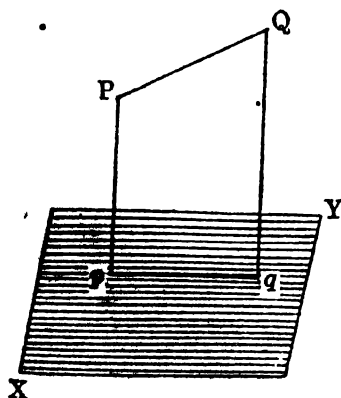


Fig. 5

The projection of a curved line on a plane may be a curved line, as in Fig. 4, or it may sometimes be a straight line

## FUNDAMENTAL CONCEPTS AND DEFINITIONS

when the curved line lies on a plane perpendicular to the plane of projection.

1'12. The *angle between a straight line and a plane* is measured by the angle between the straight line and its projection on the plane.

Let the straight line  $PQ$  (Fig. 6) and its projection  $pq$  (on the plane  $XY$ ) produced if necessary meet at the point  $R$  in the plane  $XY$  (as shown in the adjoining figure).

Then  $\angle QRq$  is called the angle between the straight line  $PQ$  and the plane  $XY$ .

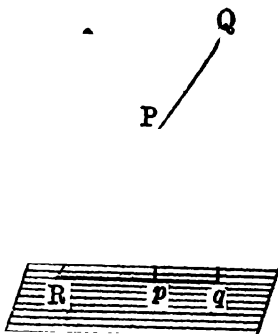


Fig. 6

1'13. The *angle between two planes* is measured by the plane angle contained by the two straight lines drawn from any point in the line of section of the two planes, perpendicular to that line of section, one in each plane.

This angle is called a **dihedral angle** between the two planes.

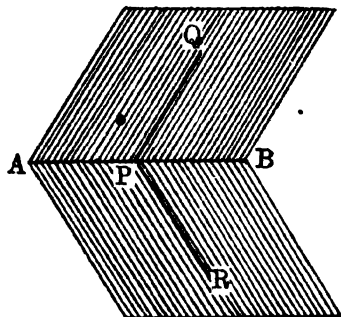


Fig. 7

Let  $\pi_1, \pi_2$  be two planes and let  $AB$  be their line of intersection. From any point  $P$  in  $AB$ , draw two straight lines  $PQ, PR$  both perpendicular to  $AB$ , one in each plane. Then the angle between the two planes is measured

## CHAPTER II

### AXIOMS AND THEOREMS

**Axioms.**—The following fundamental properties have been laid down as axioms.

(1) One and only one plane may be made to pass through any two intersecting straight lines.

(2) Two intersecting planes cut one another in a straight line and in no point outside it.

From above, the following inferences can be drawn at once :

The position of a plane is fixed if it passes through

(1) a given straight line and a point outside it.

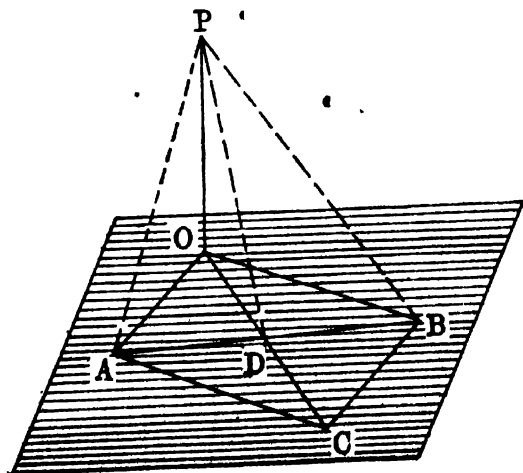
(2) two intersecting straight lines.

(3) three non-collinear points.

(4) two parallel straight lines.

#### THEOREM I

*If a straight line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane in which they lie.*



Let  $OP$  be perpendicular to each of two intersecting lines  $OA, OB$  at their point of intersection at  $O$ .

It is required to prove that  $PO$  is perpendicular to the plane  $AOB$ .

Through  $O$  draw *any* straight line  $OC$  in the plane  $AOB$ . Also in the same plane, through  $C$ , draw  $CA$  parallel to  $OB$ , and  $CB$  parallel to  $OA$ , so that  $OACB$  is a parallelogram. Join  $AB$  to meet  $OC$  in  $D$ , so that  $D$  is the mid-point of  $AB$ .

[  $\therefore$  diagonals of a parallelogram bisect each other ]

Join  $PA, PB, PD$ .

*Proof.* In the triangle  $PAB$ , the base  $AB$  is bisected at  $D$ .

$$\therefore PA^2 + PB^2 = 2AD^2 + 2PD^2 \quad \dots \quad (1)$$

Similarly from the triangle  $OAB$ , we get

$$OA^2 + OB^2 = 2AD^2 + 2OD^2 \quad \dots \quad (2)$$

$\therefore$  subtracting (2) from (1),

$$(PA^2 - OA^2) + (PB^2 - OB^2) = 2(PD^2 - OD^2) \quad (3)$$

$$\begin{aligned} \text{Now, } PA^2 - OA^2 &= OP^2; \} \\ \text{and } PB^2 - OB^2 &= OP^2 \} \end{aligned} \quad \dots \quad (4)$$

since,  $\angle POA, \angle POB$  are right angles.

$\therefore$  from (3) and (4), we get

$$2OP^2 = 2(PD^2 - OD^2).$$

$$\therefore OP^2 + OD^2 = PD^2$$

$\therefore \angle POD$  i.e.,  $\angle POC$  is a right angle.

Thus,  $PO$  is perpendicular to *any* line  $OD$  which meets it in the plane  $AOB$ .

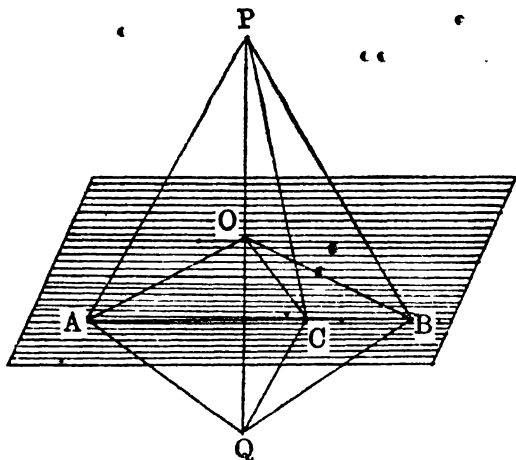
$\therefore PO$  is perpendicular to the plane  $AOB$ . Q. E. D.

*Alternative method of Proof.*

Let  $PO$  be perpendicular to both  $OA, OB$ .

It is required to prove that  $PO$  is perpendicular to the plane of  $OA, OB$  i.e. the plane  $AOB$ .

Join  $AB$  and in the plane of  $AOB$  draw *any* straight line  $OC$ , meeting  $AB$  in  $C$ . Then it will be sufficient to prove that  $PO$  is perpendicular to  $OC$ . Produce  $PO$  beyond the plane  $OAB$  to  $Q$ , making  $OQ = OP$ .



.Fig. 9

Join  $PA, PB, PC$ ; also  $QA, QB, QC$ .

Now  $AO$  being perpendicular bisector of  $PQ$ ,  $PA = QA$  and  $BO$  being perpendicular bisector of  $PQ$ ,  $PB = QB$ .

Hence,  $\triangle^s PAB, QAB$  are congruent ( $AB$  being common to both).  $\therefore \angle PAB = \angle QAB$ .

Then in  $\triangle^s PAC, QAC$ ,  $PA = QA$ ,  $AC$  is common and  $\angle PAC = \angle QAC$ .  $\therefore \triangle^s PAC, QAC$  are congruent.  $\therefore CP = CA$ .

Now in the  $\triangle^s OPC$  &  $OQC$ ,  $OP = OQ$ ,  $PC = QC$  and  $OC$  is common.  $\therefore \triangle^s OPC, OQC$  are congruent.

$\therefore \angle POC = \angle QOC$  which being adjacent angles on the straight line  $POQ$ ,  $\angle POC = \angle QOC =$  a right angle.

$\therefore PO$  is perpendicular to *any* line  $OC$  which meets it in the plane  $AOB$ .

$\therefore PO$  is perpendicular to the plane of  $OA, OB$ . Q.E.D.

### Exercise 1

1. Show that from a point in space, three straight lines can be drawn so that each is perpendicular to the plane of the other two.

2. A straight line is drawn through the centre  $O$  of a circle, perpendicular to the two radii  $OA$ ,  $OB$  of the circle. Show that all points on the circumference of the circle are equidistant from any point on the line.

[ Let  $P$  be any point on the line  $OL$ , drawn perp. to  $OA$ ,  $OB$ . Hence  $PO$  is perp. to the plane of the circle. Let  $C$  be any point on the circumference, Then  $PO$  is perp. to  $OC$ . Then, since  $OA=OC$ ,  $PO$  is common and  $\angle POA = \angle POC$ , each being a right angle.  $\therefore \triangle POA, POC$  are identically equal.  $\therefore CP=AP$ . Since  $P$  and  $C$  are any points, theorem is proved. ]

3. If  $O$  be a point in the plane of the triangle  $ABC$  and if  $P$  be a point outside the plane such that  $PO$  is perpendicular to  $OA$  and  $OB$  and if  $PA=PB=PC$ , show that  $O$  is the circum-centre of the triangle  $ABC$ .

[ Since  $PO$  is  $\perp$  to  $OA$ ,  $OB$ ,  $PO$  is perp. to the plane of the triangle  $ABC$  and hence  $\perp$  to  $OC$ . In the  $\triangle POA, POB$ , since  $PA=PB$ ,  $\angle POA = \angle POB$  each being a right angle, and  $PO$  is common,  $\triangle POA, POB$  are identically equal.  $\therefore OA=OB$ . Similarly from  $\triangle POA, POC$ ,  $OA=OC$ .  $\therefore OA=OB=OC$ . ]

4. If  $O$  be the circum-centre of any given triangle  $ABC$  and if  $P$  be any point outside the plane of the triangle  $ABC$  such that  $PA=PB=PC$ , show that  $PO$  is perpendicular to the plane  $ABC$ .

5.  $P$  is any point outside a given plane, and  $O, A, B, C, D$  are points in the plane such that  $\angle POA = \angle POB = \text{a right-angle}$ . If  $PA=PB=PC=PD$ , prove that the points  $A, B, C, D$  are concyclic. Find the centre of the circle, passing through  $A, B, C, D$ .

6.  $ABC$  is a triangle right-angled at  $C$ .  $P$  is a point outside the plane  $ABC$ , such that  $PA=PB=PC$ . If  $D$  be the mid-point of  $AB$ , prove that  $PD$  is perpendicular to  $CD$  and hence deduce that  $PD$  is perpendicular to the plane of the triangle.

[ Here  $AD=BD=CD$ .  $\therefore \triangle PDA, PDB, PDC$  are identically equal.  $\therefore \angle PDA = \angle PDB = \angle PDC$ : Now since  $\angle PDA, PDB$ , are adjacent, each is a right angle.  $\therefore \angle PDC$  is a right angle. Hence  $PD$  is perp. to  $DA, DC$  ]

## THEOREM II

*All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.*

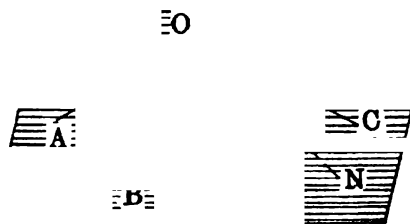


Fig. 10

Let  $PO$  be the given straight line and let the straight lines  $OA$ ,  $OB$ ,  $OC$  be drawn perpendicular to  $PO$  at  $O$ .

It is required to prove that  $OA$ ,  $OB$ ,  $OC$  are co-planar *i.e.*, they all lie in one plane.

*Proof.* Evidently  $OA$ ,  $OB$  lie in one plane and  $PO$ ,  $OC$  in another plane.

If possible, let the plane  $POC$  cut the plane  $AOB$  in the straight line  $ON$ .

Now, since  $PO$  is perpendicular to  $OA$  and  $OB$ ,  $PO$  is perpendicular to the plane  $AOB$ .

But  $ON$  being in the plane  $AOB$  and meeting  $PO$  at  $O$ ,  $PO$  is perpendicular to  $ON$ ; hence  $\angle PON$  is a right angle. Also  $\angle POC$  is a right angle.

Now,  $PO$ ,  $OC$ ,  $ON$  lying in the same plane,  $\angle POC$ ,  $\angle PON$  cannot be both right angles. Hence,  $ON$  coincides with  $OC$ . Thus,  $OC$  is the line of intersection of the planes  $AOB$  and  $POC$ . Hence  $OC$  lies in the plane of  $OA$ ,  $OB$ .

Similarly it can be easily shown that if other lines  $OD, OE, OF$  etc. are drawn perpendicular to  $PO$  at  $O$ , they lie in the plane of  $AOB$ . Hence,  $OA, OB, OC, OD, OE, OF$  etc. are all co-planar.  
Q. E. D.

### Exercise 2

1. How many horizontal straight lines can be drawn through a given point of a vertical line and how do they lie ?

How many vertical lines can be drawn through a given point ?

[ An infinite number of horizontal lines ; they all lie in one plane. One vertical line. ]

2. Through the mid-point  $O$  (i.e., the intersection of its diagonals) of a horizontal square  $ABCD$ , a vertical line  $OP$  is drawn. Show that  $PA, PB, PC, PD$  are all equal.

[ Horizontal square means a square lying on a horizontal plane.

$OP$  being perp. to the plane  $ABCD$ , is perp. to the diagonals  $AC, BD$  at their common mid-point  $O$ . Hence  $\triangle POA, POB, POC, POD$  are congruent. ]

3. Prove that there cannot be more than three mutually perpendicular straight lines in space meeting at a point.

4. If a triangle revolves about its base, show that the vertex describes a circle.

[ Let  $AM$  be drawn perp. from the vertex  $A$  to the base  $BC$ , (or  $BC$  produced). Let  $AM = p$  ; since  $p$  is constant, vertex will describe a circle of radius  $p$  and centre  $M$ . The different positions of the line  $AM$  are co-planar each being perpendicular to the base  $BC$ . ]

5. Find the locus of a point in space—

(i) equidistant from two given points ;

[ Let  $P$  and  $Q$  be two points.  $P$  and  $Q$  is joined and  $PQ$  is bisected at  $O$ . A plane is drawn through  $O$  perpendicular to  $PQ$ . Then this plane is the required locus.

Let  $A$  be any point in this plane.  $AO, AP$  and  $AQ$  are joined.

Since  $OA$  is a straight line lying in this plane and  $OQ$  is perpendicular to this plane,  $OQ$  is perpendicular to  $OA$ .

Now considering  $\triangle AOP$  and  $\triangle AOQ$ ,  $OP = OQ$  ;  $OA$  is common and



Similarly it can be proved any point in this plane is equidistant from  $P$  and  $Q$ . Therefore this point is the required locus.]

(ii) equidistant from three given non-collinear points.

[ Let  $A, B, C$  be three non-collinear points. Points in space equidistant from  $A$  and  $B$  lie on a plane perpendicular to  $AB$  and passing through mid-point of  $AB$  [ as in Ex. 5(i)]. Similarly points in space equidistant from  $B$  and  $C$  lie on a plane passing through mid-point of  $BC$  and perpendicular to it. These planes intersect each other along a line. Points upon these straight lines being common to both planes equidistant from  $A, B, C$ . Therefore the locus is the line of intersection of these two planes. ]

6. Find a point in a given straight line in space which is equidistant from two given points outside the line. When is this impossible ?

[ Let  $C$  and  $D$  be two points outside the given straight line. Consider a plane passing through the mid-point of  $CD$  say  $O$  and perpendicular to  $CD$ . Let this plane intersect  $AB$  at  $P$ .

$$\text{Therefore } PC^2 = PO^2 + OC^2 \text{ and } PD^2 = PO^2 + OD^2.$$

$$\therefore PC = PD \text{ since } OC = OD.$$

Hence  $P$  is the required point.

If  $CD$  is perpendicular to  $AB$ , then the plane passing through  $O$  perpendicular to  $CD$  becomes parallel to  $AB$ . Therefore to find out a point  $P$  becomes impossible.]

7. Prove that a point can be found in a plane equidistant from three points outside the plane. State the exceptional case, if any.

[ Let  $p$  be the given plane and  $A, B, C$  be three points outside this plane. Consider a plane  $p_1$  which passes through the mid-point of  $AB$  and perpendicular to  $AB$ . Then any point on the plane  $p_1$  is equidistant from  $A$  and  $B$ . Similarly let  $p_2$  be the plane which passes through the mid-point of  $BC$  and perpendicular to  $BC$ . Then any point on the plane  $p_2$  is equidistant from  $B$  and  $C$ . Therefore every point on the line of intersection of the two planes  $p_1$  and  $p_2$  is equidistant from  $A, B$  and  $C$ . Let this line of intersection intersect the plane  $p$  at the point  $X$ . Then  $X$  is equidistant from  $A, B, C$ .

Therefore  $X$  is the required point. If the line of intersection of the planes  $p_1$  and  $p_2$  be parallel to the plane  $p$ , no such point can be found. ]

8. Show that there is one and only one point equidistant

[ Let  $A, B, C, D$  be four points not lying in one plane and no three of which are collinear.

All points in space equidistant from  $A$  and  $B$  must lie on a plane which passes through the mid-point of  $AB$  and perpendicular to  $AB$ . Similarly the points equidistant from  $B$  and  $C$  and those from  $C$  and  $D$  respectively lie on two planes passing through the middle points  $BC$  and  $CD$  and perpendicular to  $BC$  and  $CD$  respectively. These three planes mutually intersect in three straight lines. Thus three straight lines meet at only one point common to three planes and this point is obviously equidistant from  $A, B, C, D$ .

Hence there is only one point equidistant from  $A, B, C, D$ . ]

### THEOREM III

*If two straight lines are parallel and if one of them is perpendicular to a plane, then the other is also perpendicular to the plane.*

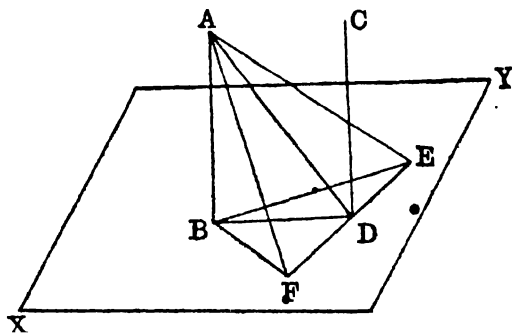


Fig. 11

Let  $AB, CD$  be two parallel straight lines of which  $AB$  is perpendicular to the plane  $XY$ .

It is required to prove that  $CD$  is also perpendicular to the plane  $XY$ .

Let  $B, D$  be the points at which the given lines meet the plane. Join  $AD, BD$ .

In the plane  $XY$ , draw the straight line  $EDF$  perpendicular to  $BD$ , making  $DE = DF$ .

Join  $AE, AF; BE, BF$ .

*Proof.* Since  $BD$  bisects  $EF$  at right angles.

$$\therefore BE = BF.$$

Now in the  $\triangle ABE, ABF$ ,  $AB$  is common and  $BE = BF$ ; also  $\angle ABE = \angle ABF$ , since each is a right angle because  $AB$  is perpendicular to the plane  $XY$  and  $BE, BF$  lie in the plane.

$\therefore \triangle ABE, ABF$  are congruent.

Hence,  $AE = AF$ .

Again in the  $\triangle ADE, ADF$ ,  $AD$  is common,  $DE = DF$  and  $AE = AF$ .

$\therefore \triangle ADE, ADF$  are congruent.

$\therefore \angle ADE = \angle ADF$  and these being adjacent angles on the straight line  $FDE$ ,  $FD$  is perpendicular to  $AD$ .

Since  $FD$  is perpendicular to  $DA$  and  $DB$ ,  $FD$  is perpendicular to the plane  $DAB$ .

As  $AD, DB$  both lie in the plane of the parallel lines  $AB, CD$ , four lines  $AB, CD, AD, BD$  are co-planar. Hence,  $CD$  lies in the plane of  $ABD$  and thus  $FD$  is perpendicular to  $CD$ .

Since  $AB, CD$  are parallel and  $BD$  meets them,

$$\therefore \angle CDB + \angle ABD = 2 \text{ right angles.}$$

But  $\angle ABD = \text{a right angle.}$

$$\therefore \angle CDB \text{ is a right angle.}$$

Hence,  $CD$  is perpendicular to  $BD$ .

Thus,  $CD$  being perpendicular to both  $DB, DF$ , is perpendicular to the plane  $XY$  in which they lie.

$\therefore CD$  is perpendicular to the plane  $XY$ .

Q. E. D.

**Cor. 1.** It can be easily proved that the converse of the theorem is true; i.e., if two straight lines are both perpendicular to a plane, then they are parallel.

Let  $AB, CD$  be both perpendicular to the plane  $XY$ . Then with the same construction, it can be proved, as before, that  $FD$  is perpendicular to  $DA$ . But  $FD$  is perpendicular to  $DB$  (construction) and is also perpendicular to  $DC$ , since  $FD$  lies in the plane  $XY$  and  $CD$  is perpendicular to the plane  $XY$  (Hyp.) Hence,  $DC$  lies in the plane of  $DB, DA$  [Theo. II]. But  $AB$  also lies in the same plane and hence  $AB, CD$  are co-planar and since  $\angle ABD = \angle CDB$ , each being a right angle,  $\angle ABD + \angle CDB = 2 \text{ right angles.}$  Hence,  $AB$  is parallel to  $CD$ .

## ~~AXIOMS AND THEOREMS~~

### Cor. 2. Theorem on three perpendiculars

If  $AB$  is perpendicular to a plane  $XY$  and if from  $B$ , the foot of the perpendicular, a line  $BC$  is drawn perpendicular to any straight line  $DE$  in the plane, then  $AC$  is also perpendicular to  $DE$ .

*Proof.* Through  $B$  draw  $FG$  in the plane  $XY$ , parallel to  $DE$ . Since  $BC$  is perp. to  $DE$ , it is also perp. to  $FG$ . Again,  $AB$  being perp. to the plane  $XY$ ,  $AB$  is perp. to  $FG$ . Thus  $FG$  being perp. to  $AB$ ,  $BC$ , is perp. to the plane  $ABC$  and hence  $DE$  being

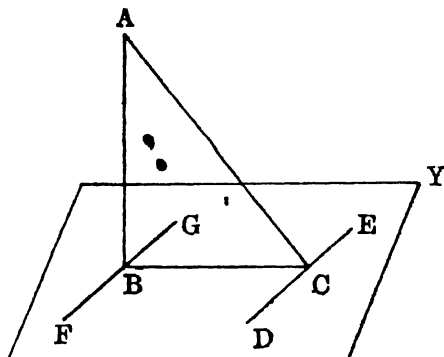


Fig. 12

parallel to  $FG$  is also perp. to the plane  $ABC$  and hence to the line  $CA$ .

Thus,  $AC$  is perpendicular to  $DE$ .

### Exercise 3

1. If the straight lines  $AB$ ,  $CD$  are perpendicular in a plane meeting it at  $B$ ,  $D$  and if  $AB$ ,  $CD$  are equal in length and on the same side of the plane, show that  $ABCD$  is a rectangle.

2. Straight lines in space which are parallel to a given straight line are parallel to one another.

[ Let the straight lines  $AB$ ,  $CD$  be each parallel to the line  $EF$ . Draw any plane  $PRQ$  perpendicular to  $EF$  meeting  $AB$  and  $CD$  at  $P$  and  $Q$  and  $EF$  at  $R$ . Then since  $AB$  is  $\parallel$  to  $EF$  and  $EF$  is perpendicular to the plane

$PRQ$ ,  $\therefore AB$  is also perpendicular to the plane  $PRQ$ . Similarly  $CD$  is perpendicular to the plane  $PRQ$ .  $\therefore AB$  and  $CD$  which are both perpendicular to the same plane  $PRQ$ , are parallel (by Cor. 1). ]

3. If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed lies in one plane and form a parallelogram.

[ Let  $P, Q, R, S$  be the middle points of the sides  $AB, BC, CD, DA$  of a skew quadrilateral.  $PQ, QR, RS, SA$  and  $BD$  are joined. Since  $P$  and  $S$  are mid-points of  $AB$  and  $CD$ ,  $PS$  is parallel and half of  $BD$ . (Note  $PS$  and  $BD$  lie in the same plane  $BAD$ .) Similarly  $QR$  is parallel and half of  $BD$ .  $\therefore PS$  and  $QR$  are parallel. Hence there is a plane passing through  $PS$  and  $QR$ .

Similarly  $PQ$  is parallel to  $RS$  and therefore there is a plane which passes through  $PQ$  and  $RS$ . But the two different planes cannot pass through four points  $P, Q, R, S$  at a time. Hence two planes must be same. In a plane,  $PS$  is equal and parallel to  $QR$ , hence  $PQRS$  is a parallelogram. ]

4.  $P$  is a point outside the plane of two parallel straight lines  $AB, CD$ . From the point  $P$ ,  $PL$  is drawn perpendicular to  $AB$  and  $LM$  is drawn perpendicular to  $CD$ . Prove that  $PM$  is perpendicular to  $CD$ .

[ Since  $AB$  and  $CD$  are parallel, therefore  $LM$  is perpendicular to both  $AB$  and  $CD$ . Consider a plane through  $LM$  and  $P$ . Since this plane is perpendicular to the plane containing  $AB$  and  $CD$ , therefore  $PM$  is perpendicular to  $CD$ . ]

5. If  $AB, CD, EF$  are three equal, parallel straight lines not lying in one plane, and if their extremities form two triangles  $ACE, BDF$ , show that the triangles are congruent.

6. If perpendiculars are drawn from a point to a system of parallel straight lines in space, show that they lie on a plane perpendicular to the parallel lines.

[ Let  $AB, CD, EF, \dots$  etc. be a system of parallel straight lines lying in the same plane  $p$  and  $OP, OQ, OR, \dots$  be respectively drawn perpendiculars to them from an external point  $O$ . Let  $XY$  be drawn parallel to  $AB, CD, EF, \dots$  through  $O$ . Since  $AB$  is parallel to  $XY$  and  $OP$  is perpendicular to  $AB$ ,  $OP$  is perpendicular to  $XY$ . Similarly  $OQ, OR, \dots$  etc. are perpendicular to  $XY$  at  $O$ . Therefore the perpendiculars  $OP, OQ, OR, \dots$  etc. must lie in the same plane and that plane will be perpendicular to  $XY$  i.e.  $AB, CD, EF, \dots$  etc. ]

CHAPTER III  
VOLUMES AND SURFACE AREAS  
OF  
REGULAR SOLIDS

**3.1.** When any portion of space is bounded by one or more surfaces, it is called a *solid figure* or simply a *solid*. These surfaces are called the faces of the solid and the intersections of the adjacent faces are called its *edges*.

When a solid is bounded by plane faces, it is called a *polyhedron*. A polyhedron is said to be *regular*, if its faces are all regular, such as equilateral triangles, squares, etc.

**3.2. Parallelopiped.**

If a solid is bounded by three pairs of parallel planes, it is called a *parallelopiped*.

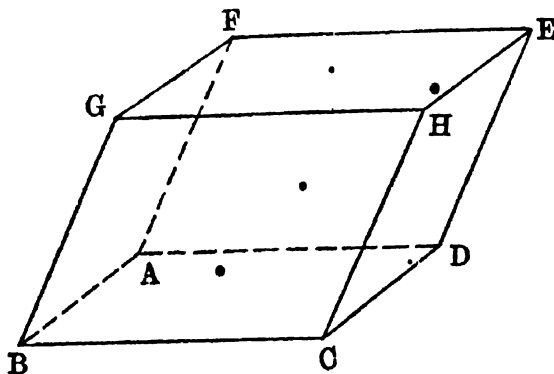


Fig. 13

Here  $ABCDEFGH$  is a parallelopiped. Its faces are all parallelograms.

**Rectangular parallelopiped.**

If a parallelopiped has its faces all rectangles, it is called a *rectangular parallelopiped* (or a *cuboid*)

Let  $OADBPQRC$  be a rectangular parallelepiped. Here  $OA$ ,  $OB$ ,  $OC$  are three mutually perpendicular lines.

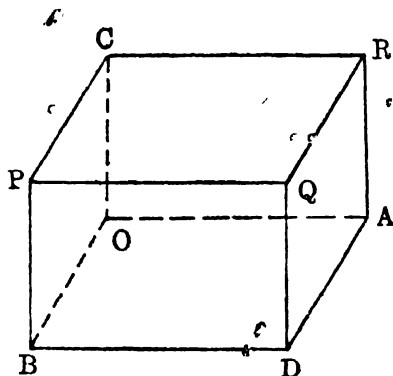


Fig. 14

Here,  $\angle COA$ ,  $\angle COB$  are right angles.

$\therefore OC$  is perpendicular to the face  $AODB$ .

$\therefore OC$  is perpendicular to  $OD$ , as  $OD$  lies in the plane  $AODB$ .

Hence,  $QD$  being parallel to  $OC$  is perpendicular to  $OD$ .

$\therefore \angle ODQ$  is a right angle.

$\therefore$  from  $\triangle ODQ$ ,  $OQ^2 = OD^2 + DQ^2 = OD^2 + OC^2$ , since  $OC = DQ$ .

Now  $\angle OAD$  being a right angle,  $OD^2 = OA^2 + AD^2 = OA^2 + OB^2$   
(as  $AD = OB$ )

$\therefore OQ^2 = OD^2 + OC^2 = OA^2 + OB^2 + OC^2$ .

Let  $OA = a$ ,  $OB = b$ ,  $OC = c$ .  $\therefore OQ^2 = a^2 + b^2 + c^2$ .

There are *three* pairs of rectangular faces parallel two by two, viz.  $(PBDQ, COAR)$ ,  $(DARQ, BOCP)$ ,  $(CPQR, OBDA)$ . The opposite faces are congruent. The four diagonals are  $OQ$ ,  $AP$ ,  $BR$ ,  $CD$ , and they are all equal.

*Whole surface of the rectangular parallelepiped*

$$= 2(bc + ca + ab).$$

*Volume*  $= abc$

i.e.,  $= \text{length} \times \text{breadth} \times \text{height}$

or,  $= (\text{area of the base}) \times \text{height}.$

### 3.3. Cube.

If all the sides of a rectangular parallelopiped are equal (*i.e.*, if the bounding faces are all squares) then the parallelopiped is called a *cube*.

If  $a$  denote each side or edge of a cube, then

the whole surface area of the cube  $= 6a^2$

and the volume  $= a^3$  *i.e.*,  $= (\text{edge})^3$ .

### 3.4. Prism.

A solid bounded by plane faces of which the side-faces are parallelograms, and the two end-faces called the *ends* are two congruent parallel plane polygons, is called a *prism*.

The straight lines in which the side-faces intersect two by two are called the *side-edges* of the prism. The side-faces being all parallelograms, the side-edges are all parallel and equal and the number of side-faces is equal to the number of the sides of the polygon at the end of the prism. If the two ends of a prism are polygons, it is called a polygonal prism; *e.g.*, if the two ends are triangles it is called a *triangular prism*.

#### Right Prism.

A solid bounded by plane faces of which the side-faces are rectangles, and the two end-faces are two congruent parallel plane polygons, is called a *right prism*.

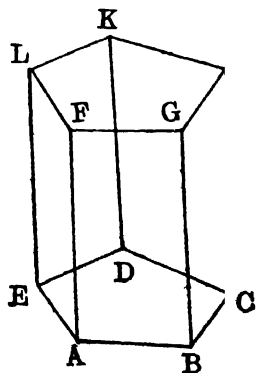
For a right prism, the side-edges are perpendicular to its ends and its height is equal to its edge. The height of a right prism is sometimes called its length. The adjoining figure is that of a right prism.

(1) *Area of the lateral surface of a right prism*

$= (\text{perimeter of the base}) \times \text{height.}$

(2) *Volume of a right prism*

$= (\text{area of the base}) \times \text{height}$





### 3.5. Pyramid.

A solid bounded by plane faces of which one, called the *base*, is any plane polygon, and the remaining faces are all triangles meeting in a point, is called a *pyramid*, the common point of the triangular faces being called its *vertex*. Obviously if the base polygon of a pyramid has  $n$  sides, the pyramid has  $n$  triangular faces. A pyramid is called a *square pyramid* or *triangular pyramid* according as the base is a *square* or a *triangle*. The *height* of a pyramid is the perpendicular distance from the vertex to the base. The straight lines in which the triangular faces intersect two by two are called its *edges* (or *slant-edges*).

#### Right Pyramid.

When a solid is bounded by plane faces of which one called the *base* is a *regular polygon*, and the remaining faces are all isosceles triangles meeting in a point (called the *vertex*), which lies on the straight line drawn perpendicular to the base from its centre (*i.e.*, the centre of the inscribed or circumscribed circle of the polygon), it is called a *right pyramid*.

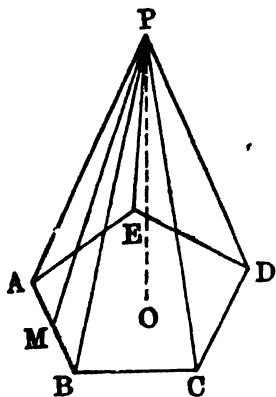


Fig. 16

For a right pyramid the edges of the base are all equal and the side-faces are equal isosceles triangles. The *slant height* of a right pyramid is the length of the perpendicular drawn from the vertex to any side of the base and hence bisecting it. Slant height is the same for each slant face. The *slant surface* of a right pyramid is the sum of its triangular faces. The *height* of a right pyramid is the length of the line joining the vertex to the centre of the base.

The adjoining figure is that of a right pyramid.

The base is  $ABCDE$ , a regular pentagon here, of which  $O$  is the centre.  $P$  is the vertex,  $PO$  is the height. It has five triangular faces which are all equal.  $PM$  is the slant height, bisecting  $AB$  at right angles. The above pyramid is very often written as  $(P-ABCDE)$ .

- (i) *Slant surface of a right pyramid*  

$$= \frac{1}{2}(\text{perimeter of the base}) \times \text{slant height}.$$

- (ii) *Volume of a right pyramid*  

$$= \frac{1}{3}(\text{area of the base}) \times \text{height}.$$

**Note.** The whole surface = slant surface + area of the base.

### 3'6. Tetrahedron.

A solid bounded by four triangular faces is called a *tetrahedron*. One of the triangular faces being taken as the base, the point where the other three meet is called the *vertex* of the tetrahedron. If all the four faces of a tetrahedron are equal equilateral triangles, the tetrahedron is said to be *regular*. All the edges of a regular tetrahedron are equal. A tetrahedron is thus a *triangular pyramid*.

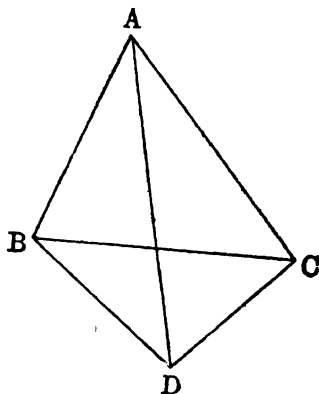


Fig. 17

The adjoining figure is a tetrahedron. It has four triangular faces,  $ABC$ ,  $ACD$ ,  $ADB$ ,  $BCD$  and six sides or edges  $AB$ ,  $BD$ ,  $DC$ ,  $CA$ ,  $AD$ ,  $BC$ . The pairs  $(AB, CD)$ ,  $(BD, AC)$ ,  $(AD, BC)$  are called *opposite edges*.  $A$  is the vertex and  $BCD$  is the base. The length of the perpendicular from the vertex  $A$  upon the face  $BCD$  is called its *height*.

- (i) *Whole surface of the tetrahedron*  

$$= \text{sum of the areas of the four faces}.$$

- (ii) *Volume of the tetrahedron*  

$$= \frac{1}{3}(\text{area of the base}) \times \text{height}.$$

### 3'7. Right circular cone.

If a solid is generated by the complete revolution of a right-angled triangle about one of the sides containing the right angle as axis the solid is called a right circular cone.

The figure (18) is that of a right circular cone. It is generated by the complete revolution of the right-angled triangle  $AOB$ , about the side  $AO$  as axis. The hypotenuse  $AB$  generates the curved surface of the cone and is called the generating line or the generator of the surface.  $AB$  is also called *slant height* of the cone and is usually denoted by  $l$ . The other side  $OB$  describes the circle  $BDCE$  of radius  $OB$  and centre  $O$ , which is called the base of the cone. The radius of the base,  $OB$ , is usually denoted by  $r$ . The point  $A$  is called the *vertex* of the cone and the length of its axis  $AO$  is called its *height* and is usually denoted by  $h$ . The angle  $\angle BAC$  is called the *vertical angle* of the cone, and  $\angle OAC$  or  $\angle OAB$  is called the *semi-vertical angle* of the cone.

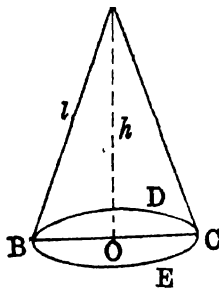


Fig. 18

If  $h$  be the height,  $r$  the radius of the base,  $l$  the slant height and  $\alpha$  the semi-vertical angle of the cone,

(i) *Area of the curved surface (or lateral surface) of a right circular cone,*

$$= \frac{1}{2}(\text{circumference of the base}) \times \text{slant height}$$

$$= \frac{1}{2} \times 2\pi r \times l = \pi r l$$

$$= \pi r \sqrt{h^2 + r^2} = \pi r^2 \operatorname{cosec} \alpha.$$

(ii) *whole surface*  $= \pi r (l + r)$ .

(iii) *volume*  $= \frac{1}{3}(\text{area of the base}) \times \text{height}$

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \tan^2 \alpha.$$

**Note.** We easily have from the right-angled triangle  $AOB$

$$r = h \tan \alpha; l = r \operatorname{cosec} \alpha. \quad l = \sqrt{h^2 + r^2}.$$

Also if  $v_1, v_2$  be the volumes of two right circular cones of heights  $h_1, h_2$  and if they have the same vertical angle  $2\alpha$ ,

$$\text{then, } v_1 : v_2 = \frac{1}{3}\pi h_1^3 \tan^2 \alpha : \frac{1}{3}\pi h_2^3 \tan^2 \alpha = h_1^3 : h_2^3.$$

*∴ the volumes of cones with the same vertical angle are to one another*

### 3.8. Right circular cylinder.

The solid generated by the complete revolution of a rectangle about one of its sides as axis is called a *right circular cylinder*.

The adjoining figure is that of a right circular cylinder. It is generated by the revolution of the rectangle  $OABC$  about the side  $AO$ .  $AO$  is called the *axis* of the cylinder. The opposite side  $BC$  generates the curved surface of the cylinder and is called the *generating line* of the cylinder. The side  $AB$  describes a *circular end* with  $A$  as centre and  $AB$  as radius and the opposite side  $OC$  describes a *circular end* with  $O$  as centre and  $OC$  as radius. Both the circular ends are called *bases* and they are equal in area. The length of the axis  $OA$  is called the *height* of the cylinder.  $OA$  is also sometimes called the length of the cylinder. If  $r$  be the radius of the base and  $h$  the height of a right circular cylinder, then

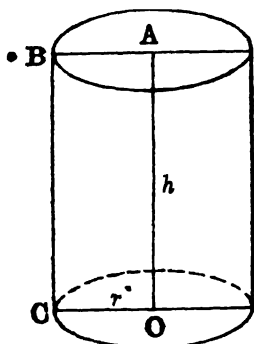


Fig. 19

(i) *area of the curved surface* .

$$= (\text{circumference of the base}) \times \text{height}$$

$$= 2\pi rh.$$

(ii) *area of the whole surface*

$$= \text{area of the curved surface} + \text{area of the two ends}$$

$$= 2\pi rh + 2\pi r^2$$

$$= 2\pi r(h + r).$$

(iii) *volume* = (area of the base  $\times$  height)

$$= \pi r^2 h.$$

### 3.9. Sphere.

If a solid is generated by the revolution of a semi-circle about its diameter as axis, it is called a *sphere*.

The figure (20) is that of a sphere generated by the complete revolution of the semi-circle  $APB$  about the diameter

$AB$  as axis and the semi-circumference  $APB$  describes the surface of the sphere.  $O$  is the centre of the semi-circle and  $OP$  its radius.

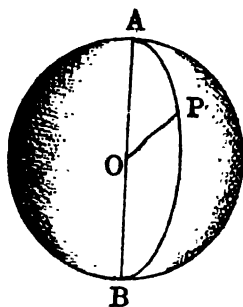


Fig. 20

As the semi-circumference revolves, all the points in it remain at a constant distance from the centre  $O$ .

Thus, a sphere may be defined as the locus of a point which moves in space in such a way that its distance from a fixed point remains constant.

The fixed point  $O$  is called the centre of the sphere and the constant distance  $OP$  is called its radius.

If  $r$  be the radius of the sphere,

$$\begin{aligned} \text{area of the surface of the sphere} &= 4\pi r^2 \\ \text{volume of the sphere} &= \frac{4}{3}\pi r^3. \end{aligned}$$

**Note.** Half of a sphere is called a hemisphere.

### 3.10. Illustrative Examples.

**Ex. 1.** If the area of the four walls of a room is 1024 sq. ft. and the height is 16 feet, find the perimeter of the floor.

Let  $a, b, c$  be the length, breadth and height of the room. Then  $c = 16$  ft.

Then,  $2c(a+b) = 1024$  sq. ft.,

$$\text{i.e., } 2(a+b) = \frac{1024}{c} = \frac{1024}{16} \text{ ft.} = 64.$$

$\therefore$  the perimeter of the floor = 64 ft.

**Ex. 2.** Find the area of the lateral surface of a right prism whose ends are squares of sides of length 3 inches and whose height is 1 ft. [C. U.]

$$\begin{aligned} \text{Area of the lateral surface} &= (\text{perimeter of the base}) \times \text{height} \\ &= 1 \text{ ft.} \times 1 \text{ ft.} = 1 \text{ sq. ft.} \end{aligned}$$

**Ex. 3.** Find the volume of the right pyramid in which the base is a triangle whose sides are 8 cm., 15 cm., 17 cm., and the height is 12 cm.

Since,  $8^2 + 15^2 = 17^2$ , the triangle is a right-angled triangle, of which the hypotenuse is of length 17 c.m.

$$\therefore \text{area of the triangle} = \frac{1}{2} \times 8 \times 15 = 60 \text{ sq. cm.}$$

$$\begin{aligned} \text{Hence, the volume of the pyramid} &= \frac{1}{3}(\text{area of the base}) \times \text{height} \\ &= \frac{1}{3} \times 60 \times 12 = 240 \text{ cu. cm.} \end{aligned}$$

**Ex. 4.** Find the whole surface of a regular tetrahedron, the length of each edge of which is  $2a$ .

Let  $ABCD$  be the tetrahedron of which  $A$  is the vertex and  $BCD$  is the base. Since  $BCD$  is an equilateral triangle of side  $2a$ , its area =  $\sqrt{3a \times a \times a}$  [ by using the formula for the area of a triangle

$$= \sqrt{s(s-a)(s-b)(s-c)} ] = \sqrt{3a^2} ]$$

$\therefore$  whole surface = sum of the areas of the four faces, which are here all equal, being equal to the area of the base  $BCD = 4 \times \sqrt{3}a^2 = 4a^2\sqrt{3}$ .

**Ex. 5.** Find the area of the curved surface and the volume of a right circular cone which is 15 ft. high and the radius of whose base is 8 ft.

[ C. U. ]

Let  $l$  be the slant height of the cone ; then  $l = \sqrt{15^2 + 8^2} = 17$  ft. Area of the curved surface of the cone =  $\pi rl$

$$= 3.1416 \times 8 \times 17 = 427.3 \text{ sq. ft. (approx.).}$$

$$\begin{aligned} \text{Volume of the cone} &= \frac{1}{3}\pi r^2 h, \text{ where } h \text{ is the height of the cone} \\ &= \frac{1}{3} \times 3.14 \times 8^2 \times 15 \\ &= 1005.31 \text{ cu. ft.} \end{aligned}$$

**Ex. 6.** The curved surface of a right circular cylinder is 2000 sq. cm. and the diameter of the base is 40 cm. ; find the volume and the height of the cylinder.

If  $r$  be the radius of the base and  $h$  the height of the right circular cylinder, area of the curved surface =  $2\pi rh = 2000$ , and  $2r = 40$ .

$$\therefore r = 20. \quad \therefore \pi h = 500 \text{ cm.}$$

$$\text{Now, volume } V = \pi hr^2 = 50 \times 400 = 20,000 \text{ cu. cm.}$$

**Ex. 7.** How many solid spheres, each 6 cm. in diameter, could be moulded from a solid metal right circular cylinder whose height is 45 cm. and diameter 4 cm. ? [ C. U. ]

$$\text{Let } r_1 \text{ be the radius of the sphere ; then its volume} = \frac{4}{3}\pi r_1^3.$$

$$\text{Here } r = 2 \text{ cm. } \therefore \text{ volume of the sphere} = \frac{4}{3}\pi \cdot 2^3.$$

Let  $r_2$  be the radius and  $h$  the height of the cylinder; then its volume  
 $= \pi r_2^2 h = \pi \times 2^2 \times 45$  (since  $r_2 = 2$ ,  $h = 45$ ).

Let  $n$  be the required number of solid spheres moulded.

$$\therefore n \times \frac{4}{3}\pi \times 3^3 = \pi \times 2^2 \times 45.$$

$$\therefore n = 5.$$

Thus, 5 spheres can be moulded,

### Examples on Chapter III

#### Sec. A : ( On rectangular parallepipeds )

1. The whole surface of a rectangular block is 52 sq. ft. The base contains 6 sq. ft., and one vertical face contains 12 sq. ft. Find the height of the block.

2. The perimeter of the floor of a room is 24 cm., and the total area of the four walls is 144 sq. cm. Find the height of the room.

3. The length, breadth and height of a rectangular block are in the ratio 5 : 6 : 7; and the whole surface of the block is 1926 sq. cm. Find the height of the block.

4. The diagonal of a rectangular parallelepiped is 13 cm. The area and perimeter of the floor are 12 sq. cm. and 14 cm. respectively. Find the volume of the parallelepiped.

[Let  $a$ ,  $b$ ,  $c$  be length, breadth and height respectively.

$$\text{As in § 3'2, } (13)^2 = a^2 + b^2 + c^2$$

$$\text{Also } 12 = a.b$$

$$14 = 2(a+b).$$

Solving the last two equations,  $a = 4$  cm;  $b = 3$  cm.

Hence from first equation,  $(13)^2 = (4)^2 + (3)^2 + c^2$

$$\therefore c = 12 \text{ cm.}$$

$$\text{Volume} = a.b.c = 4.3.12 \text{ cu. cm} = 144 \text{ cu. cm.}]$$

5. The length, breadth and height of a closed box are 10 cm., 9 cm., 7 cm. respectively, and the total inner surface is 262 sq. cm. If the walls of the box are uniformly thick, find the

Sec. B: ( On right Prisms )

1. The area of the lateral surface of a right prism is 80 sq. in. If the base of the prism be a square of side 4 in., find the height of the prism.

[ Perimeter of the base  $\equiv 4 \times 4$  in. = 16 in.

$\therefore 80 = 16 \times h$  [Note § 3'4], where  $h$  is the height of the prism.

$\therefore h = 5$  inches. ]

2. Show that the volume of a right prism of height  $h$  standing on an equilateral triangle of side  $a$  is  $\frac{\sqrt{3}}{4} a^2 h$ .

3. The base of a right prism is a trapezium whose parallel sides are 7 cm., and 3 cm., the distance between them being 4 cm. If the volume of the prism be 200 cu. cm., find the height of the prism.

4. Through a wooden pipe, whose cross-section is a square on a side of 4 cm., water flows uniformly at the rate of 50 cm. per sec. How long will it take to discharge 48 litres ?

5. Two prisms of equal height are such that the magnitude of the area of the base of one prism is double the perimeter of the base of the other prism. Show that the magnitude of the volume of the first prism is double the magnitude of the area of the lateral surfaces of the other prism.

[ Let  $2A$  be the area of base and  $h$  height of the first prism and  $A$  and  $h$  be perimeter and height of second prism.

$\therefore$  Volume of first prism  $= 2Ah$ .

Lateral surface of second prism  $= Ah$ .

$\therefore$  Volume of first prism  $= 2 \times Ah = 2 \times$  magnitude of area of lateral surface. ]

Sec. C: ( On right Pyramids )

1. A right pyramid of height 1" stands on a square base of side 4". Find the slant height and the slant edge.

2. Find the volume of a right pyramid 12 cm. high which stands on a rectangular base of sides 10 cm. and 8 cm.

3. A right pyramid stands on a rectangular base whose sides are 6' and 8', and the length of each slant edge is 13'. Find the



[ Let  $ABCD$  be rectangular base with  $O$  as centre and  $P$  be the vertex.

Now  $BD^2 = BC^2 + CD^2 = (6)^2 + (8)^2 = 100$ .

$$\therefore BD = 10'. \quad \therefore BO = 5'.$$

Again  $PB^2 = PO^2 + BO^2$  [  $PB$  is slant-edge ]

$$\text{or, } PO^2 = PB^2 - BO^2 = (13)^2 - (5)^2 = 144.$$

$$\therefore PO = 12 \text{ ft.}]$$

4. Find the volume of the right pyramid whose base is a triangle of which the sides are 13 ft., 14 ft., 15 ft. and whose height is 20 ft.

5. A right pyramid stands on a square base each of whose sides is 12 ft. and the slant faces are equilateral triangles. Find the height and the volume of the pyramid.

6.  $OA, OB, OC$  are three mutually perpendicular lines in space, and  $OA = a, OB = b, OC = c$ . Prove that the volume of the pyramid  $= \frac{1}{6}abc$ .

$$[ \text{Area of } \triangle AOB = \frac{1}{2} \cdot OA \cdot OB = \frac{1}{2} \cdot a \cdot b.$$

$$\text{Volume of the pyramid} = \frac{1}{3} \times \text{area of base} \times \text{height}$$

$$= \frac{1}{3} \times \frac{1}{2} \cdot ab \times c.$$

$$= \frac{1}{6} \cdot abc.]$$

7.  $OA, OB, OC$  are three mutually perpendicular lines of equal length  $a$ . Find the area of the triangle  $ABC$ .

Sec. D : ( On right circular cones )

1. Find the volume and the area of the slanting surface of a right circular cone of height 4 feet and the radius of whose base is 3 feet (  $\pi = \frac{22}{7}$  ).

2. If  $S$  be the area of the curved surface and  $\alpha$  the semi-vertical angle,  $h$  the height and  $r$  the radius of the base of a right circular cone, prove that

$$S = \frac{\pi h^2 \sin \alpha}{2} : \frac{\pi r^2}{\sin \alpha}.$$

[ Let  $\alpha$  be the semi-vertical angle of this right circular cone.

$$\therefore \frac{r}{h} = \tan \alpha, \text{ so that } r = h \tan \alpha.$$

$$\frac{r}{l} = \sin \alpha \text{ and } \frac{h}{l} = \cos \alpha \text{ where } l \text{ is the slant-height.}$$

$$\text{Again } S = \pi r l = \pi \cdot h \tan \alpha \cdot \frac{h}{\cos \alpha} = \pi h^2 \frac{\sin \alpha}{\cos^2 \alpha};$$

$$S = \frac{\pi r^2}{\sin \alpha} = \pi h^2 \frac{\sin \alpha}{\cos^2 \alpha} \quad ]$$

3. What is the length of the canvas 3 feet wide which will be required to make a conical tent which is 28 ft. high and which covers an area of 154 sq. yds. ? ( $\pi = \frac{22}{7}$ )

4. Show how to draw a plane parallel to the base of a right circular cone, so that it divides the cone into two parts of equal curved surfaces. [C. U.]

5. Show how to draw a plane parallel to the base of a right circular cone, so that it divides the cone into two parts of equal volumes.

6. A right circular cone 20 ft. high has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis, and if the radius of the base of the original cone be 4 feet, find the volume of the truncated cone. [C. U.]

7. If  $S_1, S_2$  be the curved surfaces and  $h_1, h_2$  the heights of two right circular cones with the same vertical angle, show that

$$S_1 : S_2 = h_1^2 : h_2^2.$$

[ Let  $r_1$  and  $r_2$  be the radii of the bases of two cones so that  $r_1 = h_1 \tan \alpha$  and  $r_2 = h_2 \tan \alpha$  where  $\alpha$  is the semi-vertical angle.

$$S_1 = \pi r_1 \sqrt{h_1^2 + r_1^2} = \pi h_1 \tan \alpha \sqrt{h_1^2 + h_1^2 \tan^2 \alpha}$$

$$S_2 = \pi h_2 \sqrt{h_2^2 + h_2^2 \tan^2 \alpha}.$$

$$\therefore \frac{S_1}{S_2} = \frac{h_1^2}{h_2^2} \quad ]$$

8. The upper portion of a right circular cone cut off by a plane parallel to the base is removed. If the curved surface of the remainder be  $\frac{3}{4}$  of that of the whole cone, show that the cutting plane bisects the altitude of the cone.

[ Proceeding as in Ex. 4,

$$\text{curved surface of } ABC = \pi r \sqrt{h^2 + r^2} = \pi h \tan \alpha \sqrt{h^2 + h^2 \tan^2 \alpha},$$

$$\text{curved surface of } AEF = \pi h_1 \tan \alpha \sqrt{h_1^2 + h_1^2 \tan^2 \alpha}.$$

Now from hypothesis,

curved surface of  $EFBC$

$$= \text{curved surface of } ABC - \text{curved surface of } AEF$$

$$= \frac{1}{4} \times \text{curved surface of } ABC.$$

$$\begin{aligned} \text{or, } \pi h \tan \alpha \sqrt{h^2 + r^2} \tan^2 \alpha - \pi h_1 \tan \alpha \sqrt{h_1^2 + h_1^2 \tan^2 \alpha} \\ = \frac{1}{2} \pi h \tan \alpha \sqrt{h^2 + h^2 \tan^2 \alpha}, \end{aligned}$$

$$\text{or, } h^2 - h_1^2 = \frac{1}{2} h^2$$

$$\text{or, } \frac{1}{2} h^2 = h_1^2.$$

$$\therefore \frac{h_1}{h} = \frac{1}{\sqrt{2}} \quad ]$$

Sec. E : ( On right circular cylinders )

1. If the volume of a right circular cylinder be 1980 cu. ft. and the area of its curved surface be 660 sq. ft., find the radius of the base and the height of the cylinder.

2. If the height of a right circular cylinder be 15'8 ft. and radius of the base be 4'2 ft. find the whole surface of the cylinder. (  $\pi = 3.1416$  )

$$[ \text{Area of the whole surface} = 2\pi r(h+r) \quad [ \text{Note } \S 3.8 ]$$

$$= 2 \times 3.1416 (15.8 + 4.2) \text{ sq. ft.}$$

$$= 527.79 \text{ sq. ft. } \quad ( \text{since } h = 15.8 \text{ ft., } r = 4.2 \text{ ft.} )$$

3. Find the height and the volume of the cylinder, the curved surface of which is 2000 sq. ft. and the diameter of whose base is 20 ft.

$$[ \text{Given } \frac{1}{\pi} = 0.31831 ]$$

4. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

5. A right prism stands on a square base whose side is 7'2 cm. Find the volume of a right circular cylinder whose height is 3 cm. and whose base touches the four sides of the square, the centre of the base being at the centre of the square.

[ Let  $ABCD$  be the square with the centre at  $O$  and the base of the right circular cylinder touches the four sides of this square with centre at same point  $O$ . Let  $E$  be one of the points at which square touches the circle.

$$AC^2 = AB^2 + BC^2 = (7.2)^2 + (7.2)^2.$$

$$\therefore AC = 7.2 \times \sqrt{2},$$

$$\therefore AO = OC = 3.6 \times \sqrt{2}.$$

$$\text{Now } OE^2 = OC^2 - CE^2 = (3.6 \times \sqrt{2})^2 - (3.6)^2 = (3.6)^2.$$

$$\therefore OE = 3.6 \text{ (radius of circular base).}$$

$$\therefore \text{Volume of cylinder} = \pi r^2 h = \pi (3.6)^2 \times 3 \text{ cu. ft.} = 122.19 \text{ cu. ft.}]$$

• Sec. F : ( On sphere )

1. Three solid golden balls of radii 3, 4 and 5 millimetres are melted into one single solid golden ball. Find the radius of the single ball. [ C. U. ]

2. A lump of clay in the form of a solid sphere is converted into a right circular cylinder of height 16 inches. Find the radius of the base of the cylinder supposing it to be equal to the radius of the sphere. [ C. U. ]

$$[ \text{Volume of solid sphere} = \frac{4}{3}\pi r^3. ]$$

$$\text{Volume of right circular cylinder} = \pi r^2 h = \pi r^2 \cdot 16.$$

$$\text{From given condition, } \frac{4}{3}\pi r^3 = \pi r^2 \cdot 16.$$

$$\therefore r = 12 \text{ inches. } ]$$

3. A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its height ? [ C. U. ]

4. The weights of two balls are in the ratio of 5 to 11 and the weights of a cubic foot of the material in the two balls are in the ratio of 121 to 25. Find the ratio of their radii.

[ From given condition

$$\frac{\frac{4}{3}\pi r^3 \cdot m}{\frac{4}{3}\pi R^3 \cdot m_1} = \frac{5}{11} \text{ where } r \text{ and } R \text{ are radii of two balls and } m \text{ and } m_1$$

are masses of a cubic foot of two balls.

$$\text{or, } \frac{\frac{4}{3}\pi r^3 \cdot 121}{\frac{4}{3}\pi R^3 \cdot 25} = \frac{5}{11} \quad \left[ \text{since } \frac{m}{m_1} = \frac{121}{25} \right]$$

$$\text{or, } \left( \frac{r}{R} \right)^3 = \left( \frac{5}{11} \right)^3 \quad \therefore \frac{r}{R} = \frac{5}{11} ]$$

5. If a solid sphere of radius 4 ft. is blown into a hollow sphere, the radius of whose external surface is 5 ft., show that the thickness of the hollow sphere, assuming it to be uniform, is approximately 1.06 ft. [ Given  $\sqrt[3]{61} = 3.94$  ]

6. The volumes of a sphere and of a right circular cylinder are as 4 : 9, and the radius of the base of the cylinder is equal to three times the radius of the sphere. Show that the radius of the sphere is three times the height of the cylinder.

[ Let  $r$  be radius of the sphere so that  $3r$  is the radius of the base of the cylinder.

From given condition,

$$\frac{\frac{4}{3}\pi r^3}{\pi(3r)^2 h} = \frac{4}{9} \text{ where } h \text{ is the height of the cylinder.}$$

$$\text{or, } \frac{\frac{4}{3}r^3}{9r^2 h} = \frac{4}{9}$$

$$\frac{r}{h} = \frac{3}{1} \quad \therefore r = 3h. ]$$

\* 7. A sphere of diameter 6 cm. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cm. If the sphere be completely submerged, by how much will the surface of the water be raised ?

\* 8. A right circular cylinder is circumscribed about a hemisphere and a right circular cone is inscribed in the cylinder, such that its vertex is at the centre of one end of the cylinder and its base coincides with the other end of the cylinder, show that

$$\frac{\text{Vol. of cone}}{1} = \frac{\text{Vol. of hemisphere}}{2} = \frac{\text{Vol. of cylinder}}{3}$$

[ Volume of the hemisphere  $\equiv V_1 = \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$  where  $r$  is the radius of the hemisphere.

Volume of the cylinder  $\equiv V_2 = \pi r^2 h$  where  $r$  and  $h$  are radius and height of the cylinder  $= \pi r^3$ . [ Since cylinder circumscribes hemisphere,  $r = h$  ]

Volume of the cone  $\equiv V_3 = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$  [  $\therefore r = h$  ]

$$\therefore \frac{1}{3}\pi r^3 = \frac{V_1}{1} = \frac{V_2}{2} = \frac{V_3}{3} ]$$

### ANSWERS

Sec. A :—(1) 4 ft. (2) 6 cm. (3) 21 cm. (4) 144 cu. cm. (5) 1 cm.

Sec. B :—(1) 5 inches. (3) 10 cm. (4) 1 minute.

Sec. C :—(1) 2'' 236, 3''. (2) 960 cu. cm. (3) 12 ft. (4) 560 cu. ft.

(5) 8'48 ft. ; 407'04 cu. ft. (7)  $\frac{\sqrt{3}}{2} a^3$ .

Sec. D :—(1) 37 $\frac{1}{2}$  cu. ft. ; 47 $\frac{1}{2}$  sq. ft. (3) 770 ft.

(4) the plane divides the height in the ratio  $\sqrt{2}-1 : 1$ .

(5) The plane divides the height in the ratio  $\sqrt{2}-1$ .

(6) 298 $\frac{1}{2}$  cu. ft.

Sec. E :—(1) radius = 6 ft. ; height = 17'5 ft. (2) 527'79 sq. ft.

(2) 31'8 ft. (5) 122'2 cu. cm.

Sec. F :—(1) 6 m. m. (2) 12 inches. (3) 50%. (4) 5 : 11.

(7) 1 cm.

# **Higher Secondary Syllabus of Elective Mathematics :**

## **CO-ORDINATE GEOMETRY**

**( Course for Class X )**

Rectangular Cartesian co-ordinates in a plane ; Lengths of segments ; Sections of a finite segment in a given ratio ; Area of a triangle ; Straight line.

## CONTENTS

| CHAPTER                               | PAGE   |
|---------------------------------------|--------|
| 1. Rectangular Cartesian Co-ordinates | ... 1  |
| 2. Equation and Locus                 | ... 12 |
| 3. Straight lines                     | ... 19 |

## IMPORTANT FORMULÆ & RESULTS

### H. S. Co-ordinate Geometry (Class X)

1. Distance  $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$   
Distance  $OP = \sqrt{x^2 + y^2}$ .
2. Point dividing the line joining two given points in a given ratio :  $x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$ ,  $y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$ .

Middle point  $\frac{1}{2}(x_1 + x_2)$ ,  $\frac{1}{2}(y_1 + y_2)$ .

3. Area of a triangle with given vertices

$$\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

4. General equation of a straight line

$$ax + by + c = 0 (a \text{ and } b \text{ both } \neq 0).$$

Every first degree equation in  $x, y$  represents a straight line.

5. Transfer of the origin (directions of axes remaining unchanged) from  $(0, 0)$  to  $(\alpha, \beta)$

$$x = X + \alpha, y = Y + \beta.$$

6. Straight line parallel to the  $x$ -axis :  $y = b$ .

Straight line parallel to the  $y$ -axis :  $x = a$ .

7. Equations of straight lines in standard forms :

(i) Intercept form :  $\frac{x}{a} + \frac{y}{b} = 1$ .

(ii) 'm' form :  $y = mx + c$ .

(iii) Form through a given point :

$$y - y_1 = m(x - x_1), \text{ or } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}.$$

(iv) Normal (or perpendicular) form :  $x \cos \alpha + y \sin \alpha = p$ .

(v) Two points form :  $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$ .



## 8. Point of Intersection of the two lines

$$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0 :$$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

## 9. Condition for concurrence of the three given lines

$$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0, a_3x + b_3y + c_3 = 0 :$$

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0.$$

10. Condition for collinearity of the three given points  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ , is

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

## 11. Angle between two given lines :

(i) When the lines are  $y = m_1x + c_1, y = m_2x + c_2$

$$\tan \phi = \frac{m_1 - m_2}{1 + m_1m_2}.$$

(ii) When the lines are

$$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$$

$$\tan \phi = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}.$$

## 12. Conditions for

(a) parallel lines, (i)  $m_1 = m_2$ , (ii)  $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ .

(b) perpendicular lines, (i)  $m_1m_2 = -1$ ,  
(ii)  $a_1a_2 + b_1b_2 = 0$ .

13. Length of the perpendicular from the point  $(x_1, y_1)$  upon the line  $ax + by + c = 0$  is

$$\pm \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

14. Equations of the bisectors of the angle between the lines  $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$  are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

# CO-ORDINATE GEOMETRY

( To be taught in Class X )

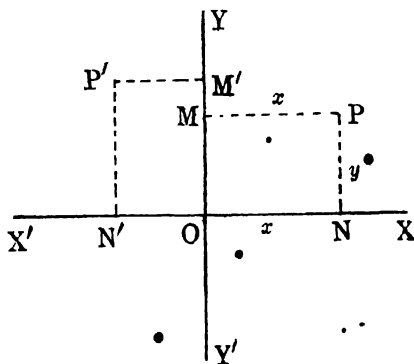
## CHAPTER I

### RECTANGULAR CARTESIAN CO-ORDINATES : ELEMENTARY RESULTS

#### 1.1. Location of a point on a plane.

##### Rectangular Cartesian Co-ordinates in a plane.

To locate the position of a point on a plane, we usually take two fixed straight lines on the plane, intersecting one another at right angles, for reference.



These lines are termed the *axes of reference* or *axes of co-ordinates*. One of them,  $XOX'$  (from right to left) is usually taken as the  $x$ -axis, and the other,  $YOY'$  (from above downwards) is taken as the  $y$ -axis. The point of intersection,  $O$ , is referred to as the *origin*. The position of a point  $P$  on the plane will be definitely known when we know its perpendicular distances  $PM$  and  $PN$  from the axes of reference. These perpendicular distances, with proper sign are termed the *co-ordinates* or more precisely the rectangular or orthogonal co-ordinates of the point. The distance  $MP$  parallel to the  $x$ -axis, of  $P$  from the  $y$ -axis is called the  $x$ -co-ordinate, or *abscissa* of  $P$ , and conventionally

measured positive from  $O$  along  $OX$  towards the right, distances from  $O$  towards  $X'$  being reckoned as negative. Similarly, the distance  $NP$  parallel to the  $y$ -axis, from the  $x$ -axis to the point, is called the  $y$ -co-ordinate or the *ordinate* of  $P$ , being conventionally measured positive upwards along  $OY$  and negative downwards along  $OY'$ .

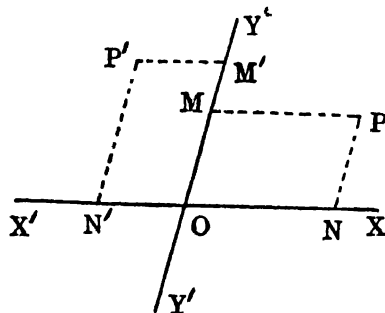
The whole plane is divided by the axes  $XOX'$  and  $YOY'$  into four quadrants  $XOY$ ,  $YOX'$ ,  $X'OY'$  and  $Y'OX'$  and according to the conventions given above, the co-ordinates of a point  $P$  are both positive in the first quadrant. In the second quadrant, (for instance for the point  $P'$ ), the  $x$ -co-ordinate is negative and the  $y$ -co-ordinate is positive. Both the co-ordinates are negative in the third quadrant, while in the fourth quadrant, the  $x$ -co-ordinate is positive but the  $y$ -co-ordinate is negative.

Generally,  $P$  being any point on the plane of axes of reference  $XOX'$  and  $YOY'$ ,  $PN$  being drawn perpendicular to the  $x$ -axis  $ON$  is the  $x$ -co-ordinate and  $NP$  is the  $y$ -co-ordinate of  $P$ , and these co-ordinates being given in magnitude and sign, the position of  $P$  is definitely fixed. Conversely, if  $P$  be given in position on the plane, its co-ordinates are definite in magnitude and sign.

The above method of locating a point on a plane is due to Des Cartes, after whom the co-ordinates are referred to as Cartesian Co-ordinates (*Rectangular* or, *Orthogonal*) in this case.

#### Oblique co-ordinates.

Instead of two mutually perpendicular lines, we may take any two intersecting lines  $XOX'$  and  $YOY'$  inclined at any angle to one another as



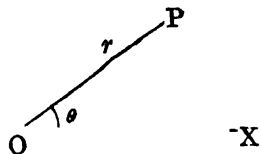
axes of reference and proceed to define the position of a point  $P$ , not by its

## ELEMENTARY RESULTS

perpendicular distances from the axes, but by the distances  $MP$  and  $NP$  of  $P$  parallel to  $X'OX$  and  $Y'OY$  respectively from the other axis. In other words, if  $PN$  be drawn parallel to  $YOY'$  to meet  $XOX'$  at  $N$ ,  $ON$  and  $NP$  are respectively the  $x$ -co-ordinate and the  $y$ -co-ordinate of  $P$ , which define the position of  $P$  on the plane definitely. The axes are here termed as *oblique axes* and the co-ordinates are *oblique co-ordinates* (Cartesian). The convention as to the signs of the co-ordinates is exactly as in the case of rectangular axes.

### Polar Co-ordinates.

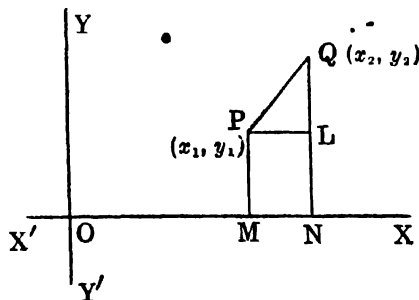
There is yet another method of locating a point on a plane. Here we take a fixed straight line  $OX$  on the plane for reference, which we call the *initial line*, and a fixed point  $O$  on it, called the *origin* or *pole*. The position of a point  $P$  is definitely known if we know the



distance  $OP$  ( $=r$  say), and the angle  $XOP$  ( $=\theta$  say). The quantities  $r$  and  $\theta$  are called the *polar co-ordinates* of  $P$ . The distance  $r$ , called the *radius vector*, is always taken as positive, and the angle  $\theta$ , called the *vectorial angle* is conventionally positive when measured anti-clockwise from  $OX$ . The polar co-ordinates being given, the position of  $P$  is definite, while if the position of  $P$  be given, its polar co-ordinates will have definite values.

Henceforth, throughout the book, we shall use rectangular Cartesian co-ordinates only to refer to positions of points on a plane.

**1.2. Lengths of Segments : Distance between two points whose Cartesian co-ordinates are given.**



Let  $P$  and  $Q$  be two given points on a plane whose Cartesian co-ordinates (rectangular) are  $(x_1, y_1)$  and  $(x_2, y_2)$  respectively.

## CO-ORDINATE GEOMETRY

Let  $PM$  and  $QN$  be perpendiculars from  $P$  and  $Q$  on the  $x$ -axis  $OX$ , and  $PL$  perpendicular from  $P$  on  $NQ$ . Then  $OM = x_1$ ,  $MP = y_1$ ,  $ON = x_2$ ,  $NQ = y_2$ .

$$\therefore PL = MN = ON - OM = x_2 - x_1$$

$$LQ = NQ - NL = NQ - MP = y_2 - y_1.$$

$\therefore$  from the right-angled triangle  $PLQ$ ,

$$PQ^2 = PL^2 + LQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Cor.** The distance  $r$  of a point  $P$  whose co-ordinates are given to be  $x, y$ , from the origin (whose co-ordinates are  $0, 0$ ) is given by

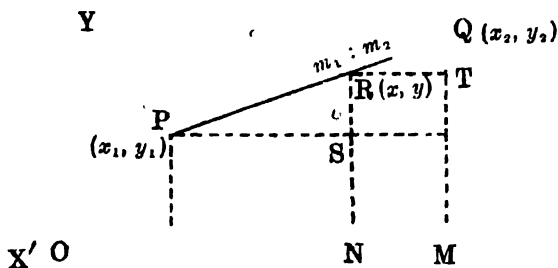
$$OP \equiv r = \sqrt{x^2 + y^2}.$$

### 1'3. Section of a given segment in a finite ratio : point dividing the line joining two given points in a given ratio.

Let  $P$  and  $Q$  be two given points whose co-ordinates are  $(x_1, y_1)$  and  $(x_2, y_2)$  respectively and let  $R$  be the point which divides  $PQ$  internally in a given ratio  $m_1 : m_2$ , i.e.,  $PR : RQ = m_1 : m_2$ .

Let  $(x, y)$  be the co-ordinates of  $R$ .

Draw  $PL$ ,  $QM$  and  $RN$  perpendiculars on the  $x$ -axis, and let  $PS$  and  $RT$  be parallel to the  $x$ -axis, meeting  $RN$  and  $QM$



at  $S$  and  $T$  respectively. Then the triangles  $PSR$  and  $RTQ$  are evidently similar.

$$\text{Hence, } \frac{PS}{RT} = \frac{SR}{TQ} = \frac{PR}{RQ} = \frac{m_1}{m_2}.$$

## ELEMENTARY RESULTS

But  $PS = \bar{ON} - OL = x - x_1$ ,  $RT = OM - ON = x_2 - x$ .

Similarly,  $SR = y - y_1$ ,  $TQ = y_2 - y$ .

$$\therefore \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} = \frac{m_1}{m_2}.$$

Hence, simplifying,

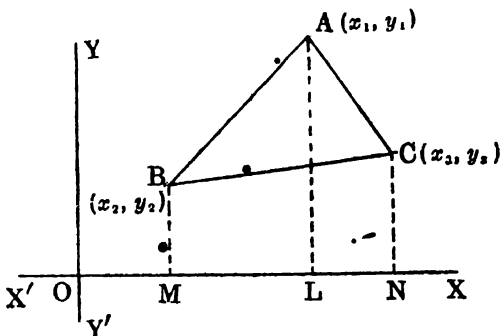
$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2},$$

giving the co-ordinates of  $R$ .

**Cor.** The co-ordinates of the middle point of the line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $\frac{1}{2}(x_1 + x_2)$ ,  $\frac{1}{2}(y_1 + y_2)$ .

**Note.** If  $R$  divides  $PQ$  *externally* in the ratio  $m_1 : m_2$ , one of the two quantities  $m_1$  and  $m_2$  is to be taken with a *negative* sign. Whichever of the two is taken negative, the co-ordinates of  $R$  will remain the same, namely  $\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}$ ,  $\frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$ .

### 1.4. Area of a triangle whose vertices are given.



Let the vertices  $A, B, C$  of a given triangle have co-ordinates  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  respectively.

Let  $AL, BM, CN$  be drawn perpendiculars on the  $x$ -axis, so that  $OL = x_1$ ,  $LA = y_1$ ,  $OM = x_2$ ,  $MB = y_2$ ,  $ON = x_3$ ,  $NC = y_3$ .

Now evidently,

$$\begin{aligned} \triangle ABC &= \text{trapezium } ABML + \text{trapezium } ALNC \\ &\quad - \text{trapezium } BMNC. \end{aligned}$$

Also, area of a trapezium =  $\frac{1}{2}$  the sum of the parallel sides  $\times$  the perpendicular distance between them.

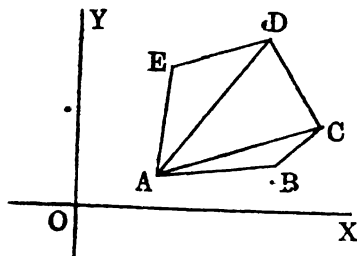
Hence, the area of the triangle  $ABC$  is given by

$$\begin{aligned}\Delta &= \frac{1}{2}(MB + LA) \times ML + \frac{1}{2}(LA + NC) \times LN \\ &\quad - \frac{1}{2}(MB + NC) \times MN \\ &= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_2 - x_1) \\ &\quad - \frac{1}{2}(y_2 + y_3)(x_3 - x_2) \\ &= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\end{aligned}$$

on simplification.

**Note 1.** In our figure above, we have taken the vertices  $A, B, C$  in such an order that in moving along the sides  $AB, BC, CA$  we are to move in an anti-clockwise direction. In such a case, the above expression for the area of the triangle  $ABC$  will always be found to be *positive*. If the vertices in order are so taken that in moving along the sides of the triangle in that order we move clockwise, the above expression for the area will be found to be *negative*.

**Note 2.** Area of a polygon with given vertices.



If a polygon  $ABCDE$  has the co-ordinates of its vertices given, to find its area, we break it up into triangles by joining its diagonals, and then add up the areas of these triangles (which have got their vertices known) in the same order. The result will give the required area of the polygon in that order.

### 1.5. Illustrative Examples.

**Ex. 1.** Prove that  $(-1, -1)$ ,  $(1, 1)$  and  $(-\sqrt{3}, \sqrt{3})$  are the vertices of an equilateral triangle.

Let  $A, B, C$  be the points whose co-ordinates are  $(-1, -1)$ ,  $(1, 1)$  and  $(-\sqrt{3}, \sqrt{3})$  respectively.

$$\begin{aligned}\text{Then } AB^2 &= (-1-1)^2 + (-1-1)^2 = 8 \\ BC^2 &= (1+\sqrt{3})^2 + (1-\sqrt{3})^2 = 8 \\ CA^2 &= (-\sqrt{3}+1)^2 + (\sqrt{3}+1)^2 = 8\end{aligned}$$

$$\therefore AB^2 = BC^2 = CA^2 \text{ or } AB = BC = CA.$$

Thus the triangle  $ABC$  is equilateral.

**Ex. 2.** If the co-ordinates of  $A, B, C, D$  are  $(-1, -2), (7, 4), (4, 8)$  and  $(-4, 2)$  respectively, show that  $ABCD$  is a rectangle.

$$\text{Here, } AB^2 = (-1-7)^2 + (-2-4)^2 = 100$$

$$BC^2 = (7-4)^2 + (4-8)^2 = 25$$

$$CD^2 = (4+4)^2 + (8-2)^2 = 100$$

$$DA^2 = (-4+1)^2 + (2+2)^2 = 25$$

$$\text{and } AC^2 = (-1-4)^2 + (-2-8)^2 = 125.$$

$$\text{Thus, } AB = CD \text{ and } BC = DA.$$

Hence, opposite sides being equal,  $ABCD$  must be a parallelogram.

Moreover, here,  $AC^2 = AB^2 + BC^2$ , so that angle  $ABC$  is a right angle.

Hence,  $ABCD$  is a rectangle.

**Ex. 3.** Find the circum-centre and the circum-radius of the triangle whose vertices are the points  $(-4, 3), (-2, -3), (0, -5)$ .

Let  $A, B, C$  be the vertices of the triangle whose co-ordinates are  $(-4, 3), (-2, -3), (0, -5)$  respectively, and let  $S$  be the circum-centre of the triangle, whose co-ordinates are supposed to be  $(x, y)$ .

$$\text{Then, } SA = SB = SC \text{ or } SA^2 = SB^2 = SC^2.$$

$$\therefore (x+4)^2 + (y-3)^2 = (x+2)^2 + (y+3)^2 = (x-0)^2 + (y+5)^2,$$

$$\text{or, } 8x - 6y + 25 = 4x + 6y + 13 = 10y + 25,$$

whence solving,  $x = 6, y = 3$ .

$\therefore$  The circum-centre  $S$  is the point  $(6, 3)$ .

Also, circum-radius  $= SA = \sqrt{(6+4)^2 + (3-3)^2} = 10$  units.

**Ex. 4.**  $A, B, C, D$  are four points whose co-ordinates are  $(1, -8), (-3, 4), (3, 7)$  and  $(3, 16)$  respectively. Find the ratio in which  $CD$  divides  $AB$ . Find also the ratio in which  $AB$  divides  $CD$ .

Let  $AB$  and  $CD$  intersect each other at  $P$  which divides  $AB$  in the ratio  $\lambda : 1$ .

Then, the co-ordinates of  $P$  (by § 1.3) are

$$\frac{-9\lambda + 1}{\lambda + 1}, \frac{4\lambda - 8}{\lambda + 1} \text{ respectively.}$$



Now,  $P, C, D$  being in one straight line, the area of the triangle  $PCD$  is zero. Hence (by § 14).

$$\frac{1}{2} \left[ \frac{-3\lambda+1}{\lambda+1} (16-15) + 0 \left( 16 - \frac{4\lambda-8}{\lambda+1} \right) + 3 \left( \frac{4\lambda-8}{\lambda+1} - 7 \right) \right] = 0,$$

$$\text{or, } -9(-3\lambda+1) + 3(-3\lambda-15) = 0; \therefore \lambda = 3.$$

Hence,  $CD$  divides  $AB$  in the ratio  $3 : 1$ .

The co-ordinates of  $P$  are therefore,

$$\frac{3(-3)+1}{3+1}, \frac{3 \cdot 4-8}{3+1} \text{ i.e., } -2, 1 \text{ respectively.}$$

Again, if  $P$  divides  $CD$  in the ratio  $\mu : 1$ , then co-ordinates of  $P$  are  $\frac{\mu \cdot 3+0}{\mu+1}, \frac{\mu \cdot 16+7 \cdot 1}{\mu+1}$  and these being identical with  $-2, 1$ , we have

$$\frac{3\mu}{\mu+1} = -2 \text{ and } \frac{16\mu+7}{\mu+1} = 1.$$

Either of these give  $\mu = -\frac{2}{3}$ .

Hence,  $AB$  divides  $CD$  externally in the ratio  $2 : 5$ .

**Ex. 5.** The vertices  $A, B, C$  of a triangle have co-ordinates  $(5, 6), (-9, 1)$  and  $(-3, -1)$  respectively. Find the length of the perpendicular from  $A$  on  $BC$ .

The area of the triangle  $ABC$  is

$$\frac{1}{2} [5(1+1) - 9(-1-6) - 3(6-1)] = 29 \text{ square units.}$$

$$\text{Also, distance } BC = \sqrt{(-9+3)^2 + (1+1)^2} = \sqrt{40} \text{ units.}$$

Hence, if  $p$  be the length of the perpendicular from  $A$  on  $BC$ , area of the triangle  $ABC = \frac{1}{2} \cdot p \cdot \sqrt{40} = 29$ .

$$\therefore p = \frac{29 \times 2}{\sqrt{40}} = \frac{29}{\sqrt{10}} = \frac{29}{10} \sqrt{10} \text{ units.}$$

### Examples I

1. Find the distance between the following pair of points :

(i)  $(2, 1)$  and  $(8, 9)$ .

(ii)  $(-3, 5)$  and  $(9, -2)$ .

(iii)  $(-6, -7)$  and  $(4, -9)$ .

(iv)  $(a \cos \alpha, a \sin \alpha)$  and  $(a \cos \beta, a \sin \beta)$ .

(v)  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$ .

2. Prove that the point  $(-2, -11)$  is equidistant from  $(-3, 7)$  and  $(4, 6)$ .

3. Show that the triangle whose vertices are  $(-2, -5)$   $(4, -1)$ ,  $(-1, 0)$  is isosceles.

4. If the ordinate of a point equidistant from  $(-3, 7)$  and  $(6, -11)$  be 9, find its abscissa.

5. Prove that the points  $(4, 8)$ ,  $(4, 12)$  and  $(4 + 2\sqrt{3}, 10)$  are the vertices of an equilateral triangle.

6. Show that the lines joining the point  $(4, 5)$  to the points  $(-2, -3)$  and  $(16, -4)$  are at right angles.

7. Prove that the points  $(-2, 3)$ ,  $(-3, 10)$  and  $(4, 11)$  are the angular points of an isosceles right-angled triangle.

8. Show that the lines joining successively the points  $(-2, -5)$ ,  $(7, -1)$ ,  $(8, 6)$  and  $(-1, 2)$  form a parallelogram.

9. Prove that the points  $(2, -2)$ ,  $(8, 4)$ ,  $(5, 7)$  and  $(-1, 1)$  are the successive angular points of a rectangle.

10. Show that (i) the points  $(3, -5)$ ,  $(9, 10)$ ,  $(3, 25)$  and  $(-3, 10)$  are the vertices of a rhombus;

(ii) the points  $(-a, -a)$ ,  $(a, 0)$ ,  $(0, 2a)$ ,  $(-2a, a)$  are the vertices of a square.

11. (i) Prove that the point  $(11, 2)$  is the circum-centre of the triangle whose vertices are  $(1, 2)$ ,  $(3, -4)$  and  $(5, -6)$ , and find the circum-radius of the triangle.

(ii) Find the circum-centre of the triangle whose vertices are  $(1, 1)$ ,  $(2, 3)$  and  $(-2, 2)$ .

12.  $A$  and  $B$  are two fixed points whose co-ordinates are  $(2, 4)$  and  $(2, 6)$  respectively;  $ABP$  is an equilateral triangle on the side of  $AB$  opposite to the origin. Find the co-ordinates of  $P$ .

[ H. S. 1961 ]

13. In a triangle  $ABC$ ,  $AD$  is the median bisecting  $BC$ . Prove analytically that  $AB^2 + AC^2 = 2AD^2 + 2BD^2$ .

[ Choose rectangular axes with  $A$  as origin. ]

14.  $G$  is the centroid of a triangle  $ABC$ , and  $P$  is any other point on the plane. Prove that

$$(i) \quad BC^2 + CA^2 + AB^2 = 3(GA^2 + GB^2 + GC^2).$$

$$(ii) \quad PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2.$$

[ Choose rectangular axes with  $G$  as origin. ]

15. Find the co-ordinates of the points which divide the line joining the points  $(1, -2)$  and  $(4, 1)$  in the ratio  $2:1$  (i) internally, (ii) externally.

16. The co-ordinates of the points  $A, B, C, D$  are  $(-2, 3)$ ,  $(1, -2)$ ,  $(8, -3)$  and  $(5, 2)$  respectively. Show that  $AC$  and  $BD$  bisect each other.

17. Find the areas of the triangles whose vertices are respectively,

(i)  $(3, -4), (7, 5), (-1, 10)$ .

(ii)  $(0, 4), (3, 6), (-8, -2)$ .

(iii)  $(5a, 0), (-2a, 7a), (-6a, -3a)$ .

(iv)  $(a, b+c), (a, b-c), (-a, c)$ .

(v)  $(a \cos \alpha, b \sin \alpha), (a \cos \beta, b \sin \beta), (a \cos \gamma, b \sin \gamma)$ .

$$\begin{aligned} & \left[ \frac{1}{2} \{ a \cos \alpha (b \sin \beta - b \sin \gamma) + a \cos \beta (b \sin \gamma - b \sin \alpha) \right. \\ & \quad \left. + a \cos \gamma (b \sin \alpha - b \sin \beta) \} \right] \\ &= \frac{1}{2} ab \{ \cos \alpha (\sin \beta - \sin \gamma) + \cos \beta (\sin \gamma - \sin \alpha) + \cos \gamma (\sin \alpha - \sin \beta) \} \\ &= -\frac{1}{2} ab \{ \sin (\alpha - \beta) + \sin (\beta - \gamma) + \sin (\gamma - \alpha) \} \\ &= \frac{1}{2} ab \cdot 4 \sin \frac{1}{2} (\alpha - \beta) \cdot \sin \frac{1}{2} (\beta - \gamma) \sin \frac{1}{2} (\gamma - \alpha) \\ &= 2ab \sin \frac{1}{2} (\beta - \gamma) \sin \frac{1}{2} (\gamma - \alpha) \sin \frac{1}{2} (\alpha - \beta). \end{aligned}$$

18. Prove that the area of the triangle whose vertices are the points  $(at_1^2, 2at_1)$ ,  $(at_2^2, 2at_2)$  and  $(at_3^2, 2at_3)$  respectively is  $a^2(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$ .

$$\begin{aligned} & \left[ \frac{1}{2} \{ at_1^2 (2at_2 - 2at_3) + at_2^2 (2at_3 - 2at_1) + at_3^2 (2at_1 - 2at_2) \} \right] \\ &= a^2 \{ t_1^2 (t_2 - t_3) + t_2^2 (t_3 - t_1) + t_3^2 (t_1 - t_2) \} \\ &= a^2 (t_1 - t_2)(t_2 - t_3)(t_3 - t_1). \end{aligned}$$

19. Show that the points  $(1, 4)$ ,  $(3, -2)$  and  $(-3, 16)$  are collinear. [The triangle formed by them is of zero area.]

20. Show that the points  $(0, -3)$ ,  $(3, 0)$  and  $(5, 2)$  lie on a straight line.

21. (i) Find the area of the triangle whose vertices  $A, B, C$  are respectively  $(3, 4)$ ,  $(-4, 3)$  and  $(8, 6)$ . Hence, or otherwise find the length of the perpendicular from  $A$  on  $BC$ . [H. S. 1961]

(ii) The vertices  $A, B, C$  of a triangle  $ABC$  have co-ordinates  $(5, 2)$ ,  $(-9, 3)$  and  $(-3, -5)$  respectively. Find the length of the perpendicular from  $A$  on  $BC$ .

**22.** Find the area of the quadrilateral whose angular points taken in order are (2, 8), (0, -7), (8, -6) and (6, 11).

**23.** Find the area of the quadrilateral whose angular points are respectively (1, 1), (3, 4), (5, -2), (4, -7). [C. U. 1944]

[Let  $A(2, 8)$ ;  $B(0, -7)$ ;  $C(8, -6)$  and  $D(6, 11)$ .

$\therefore$  Area of  $\square ABCD$  = Area of  $\triangle ABC$  + Area of  $\triangle ACD$  =  $9 + 2\frac{1}{2} = 20\frac{1}{2}$ .]

**24.** (i) Show that the straight line joining the points (0, -1) and (15, 2) divides the line joining the points (-1, 2) and (4, -5) internally in the ratio 2 : 3.

[Let  $P$  is the point on which  $AB$  intersects  $CD$  in the ratio 1 :  $\lambda$ .

$\therefore$  Co-ordinates of  $P$  is  $\frac{-\lambda+4}{\lambda+1}, \frac{2\lambda-5}{\lambda+1}$  as in § 1'3.

Since  $P, A, B$  lie in one straight line, hence area of  $\triangle PAB = 0$ ,

$$\text{i.e., } \frac{1}{2} \left\{ \frac{-\lambda+4}{\lambda+1} (-1-2) + 15 \left( \frac{2\lambda-5}{\lambda+1} + 17 \right) \right\} = 0,$$

whence  $\lambda = \frac{2}{3}$ . Hence the result.]

(ii) If  $A, B, C, D$  are points whose co-ordinates are (-2, 3), (8, 9), (0, 4) and (3, 0) respectively, and  $AB$  and  $CD$  are joined; find the ratio of the segments into which  $AB$  is divided by  $CD$ .

[As above co-ordinates of  $P$  is  $\frac{8\lambda-2}{\lambda+1}, \frac{9\lambda+3}{\lambda+1}$ .

Since  $P, A, B$  lie in one straight line, whence area of  $\triangle ABC = 0$ ,

$$\text{i.e., } \frac{1}{2} \left\{ \frac{8\lambda-2}{\lambda+1} (-4) + 3 \left( 4 - \frac{9\lambda+3}{\lambda+1} \right) \right\} = 0,$$

whence  $\lambda = \frac{1}{4}$ . Hence the result.]

# ANSWERS

1. (i) 10. (ii)  $\sqrt{193}$ . (iii)  $2\sqrt{26}$ . (iv)  $2a \sin \frac{1}{2}(\alpha + \beta)$ .

(v)  $a(t_1 + t_2) \sqrt{(t_1 + t_2) + 4}$ . 4. 23.5. 11. (i) 10.

(ii)  $-\frac{1}{14}, \frac{3}{14}$ . 12.  $2 + \sqrt{3}$ , 5. 15. (i) 3, 0. (ii) 7, 4.

17. (i) 46 sq. units. (ii) 1 sq. unit. (iii)  $49a^2$ . (iv)  $2ac$ .

(v)  $2ab \sin \frac{1}{2}(\beta - \gamma) \sin \frac{1}{2}(\gamma - \alpha) \sin \frac{1}{2}(\alpha - \beta)$ .

21. (i)  $4\frac{1}{2}$  sq. units;  $\frac{1}{17} \sqrt{17}$ . (ii) 10.6. 22. 96 sq. units.

23.  $20\frac{1}{2}$  sq. units. 24. (ii) 11 : 47.

## CHAPTER II

### EQUATION AND LOCUS

#### 2'1. Equation and Locus.

If an equation in  $x$  and  $y$  be given, the co-ordinates of any and every point on a plane will not generally satisfy the equation. On the other hand, there are innumerable points on the plane whose co-ordinates will satisfy the given equation ; for we may choose any value of  $x$ , and substituting that value in the given equation, solve for value or values of  $y$  whereby the given equation is satisfied. We thus get definite point or points with given  $x$ , whose co-ordinates satisfy the equation. As  $x$  may be chosen arbitrarily, the number of points will be infinite. These points joined will give us a line (straight or curved) on the plane, the co-ordinates of every point of which, (and of no point outside it), will satisfy the given equation. This line is defined as the *locus* represented by the given equation.

Conversely, if a line, straight or curved, be given on a plane then if with chosen axes on the plane, a relation between  $x$  and  $y$  can be found which is satisfied by the co-ordinates of every point of the given line, and which holds for no other points except those lying on the line, then that relation between  $x$  and  $y$  is defined as the *equation* of the given line on the plane, with the chosen axes of reference.

For example, given the equation

$$x^2 + y^2 + 6x - 4y - 12 = 0,$$

we can write it in the form

$$(x + 3)^2 + (y - 2)^2 = 25,$$

$$\text{or, } \sqrt{\{x - (-3)\}^2 + \{y - 2\}^2} = 5.$$

This relation shows that the distance of the point  $(x, y)$  from the fixed point  $(-3, 2)$  is 5. [ See § 1'2 ]. Thus the different positions of the point  $(x, y)$  must be at a fixed distance 5 from the fixed point  $(-3, 2)$ . This identifies the locus of the point  $(x, y)$  to be a circle with centre  $(-3, 2)$  and radius 5. This is therefore the locus represented by the equation.

Conversely, suppose a straight line on a plane with chosen axes be given, passing through the given points (3, 2) and (-1, 5). Then if  $(x, y)$  be the co-ordinates of any arbitrary point on the line, the area of the triangle with  $(x, y)$ , (3, 2) and (-1, 5) as vertices must be zero.

$$\text{Hence, } \frac{1}{2}\{x(2-5) + 3(5-y) + (-1)(y-2)\} = 0. \quad [\text{See § 1'4}]$$

$$\text{i.e., } -3x - 4y + 17 = 0, \quad \text{or, } 3x + 4y = 17.$$

This being the relation satisfied by the co-ordinates of any and every point on the given line, it represents the equation to the given line.

## 2'2. Every first degree equation in $x, y$ must represent a straight line.

Let  $ax + by + c = 0$  ... (i) be a given equation of the first degree. Clearly, any equation of the first degree in  $x, y$  is of this type.

Take any two points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  on the locus. Any other point  $P(x_3, y_3)$  on the locus being taken, the co-ordinates of  $A, B$  and  $P$  satisfy the given equation.

$$\text{Hence, } ax_1 + by_1 + c = 0 \quad \dots \text{(ii)}$$

$$ax_2 + by_2 + c = 0 \quad \dots \text{(iii)}$$

$$\text{and } ax_3 + by_3 + c = 0 \quad \dots \text{(iv)}$$

From (ii) and (iii), by cross-multiplication,

$$\frac{a}{y_1 - y_2} = \frac{b}{x_2 - x_1} = \frac{c}{x_1y_2 - x_2y_1}.$$

$\therefore$  substituting in (iv),

$$x_3(y_1 - y_2) + y_3(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0,$$

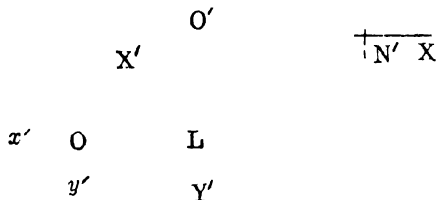
$$\text{or, } x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

This shows that the triangle  $ABP$  is of zero area [See § 1'4]. Thus,  $P$  lies on the straight line  $AB$ . As  $P$  is any point on the locus represented by (i), it follows that the locus is a straight line.

## 2'3. Transfer of origin, directions of axes remaining unchanged.

Let  $O$  be the origin, and let,  $xOx', yOy'$  be a set of rectangular axes, with respect to which co-ordinates of a point  $P$  are  $x, y$ .

Let the origin be transferred to the point  $O'$  whose co-ordinates referred to the given axes  $xOx'$  and  $yOy'$  are  $(\alpha, \beta)$ , and let the axes  $XO'X'$  and  $YO'Y'$ , which are parallel respectively to  $xOx'$  and  $yOy'$  through  $O'$  be taken as the new axes of co-ordinates. Suppose the co-ordinates of  $P$  referred to this new set of axes be  $X, Y$ .



Let  $PN$  be perpendicular to  $Ox$  and let it intersect  $XO'X'$  at  $N'$ . Also let  $YO'Y'$  intersect  $Ox$  at  $L$ .

Then evidently  $ON = x$ ,  $NP = y$ ,  $O'N' = X$ ,  $N'P = Y$ ,  $OL = \alpha$ ,  $LO' = \beta$ , in magnitude and sign.

Now, from the figure,

$$ON = OL + LN = OL + O'N'$$

$$\text{and } NP = NN' + N'P = LO' + N'P.$$

$$\text{Hence, } x = X + \alpha$$

$$y = Y + \beta.$$

These are the equations of transformation, connecting the old co-ordinates of  $P$  with the new ones.

These equations are easily seen to be true, in whichever quadrant  $O'$  may lie, provided in writing the relations from the figure, we take due notice of the signs of the quantities involved.

Thus, if an equation to a locus or curve be given referred to a set of rectangular axes, and if the origin be transferred to a position whose co-ordinates are  $\alpha, \beta$ , the axes remaining unchanged in direction, the equation of the curve referred to the

new axes will be obtained by replacing in the given equation  $x$  by  $X + \alpha$  and  $y$  by  $Y + \beta$ .

#### 2'4. Illustrative Examples.

**Ex. 1.** *Prove that the locus of a point which is always equidistant from two given points is a straight line.*

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the two given points  $A$  and  $B$ , and let  $(x, y)$  be the co-ordinates of any point  $P$  which is equidistant from them. Then,  $PA = PB$  or  $PA^2 = PB^2$ .

$$\therefore (x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2,$$

$$\text{or, } 2(x_2 - x_1)x + 2(y_2 - y_1)y + (x_1^2 + y_1^2 - x_2^2 - y_2^2) = 0.$$

This being the equation satisfied by the co-ordinates  $(x, y)$  of any position of  $P$ , it represents the equation of the locus of  $P$ .

As this is a first degree equation in  $(x, y)$ , it represents a straight line [See § 2'2]

**Ex. 2.**  *$A$  and  $B$  are two given points whose co-ordinates are  $(-5, 3)$  and  $(2, 4)$  respectively. A point  $P$  moves in such a manner that  $PA : PB = 3 : 2$ . Find the equation to the locus traced out by  $P$ . What curve does it represent?*

Let  $(x, y)$  be the co-ordinates for any position of  $P$ .

Then,  $PA = \sqrt{(x+5)^2 + (y-3)^2}$  and  $PB = \sqrt{(x-2)^2 + (y-4)^2}$ . By the given condition,  $PA : PB = 3 : 2$  or  $4PA^2 = 9PB^2$ .

$$\therefore 4\{(x+5)^2 + (y-3)^2\} = 9\{(x-2)^2 + (y-4)^2\},$$

$$\text{whence, } 5(x^2 + y^2) - 76x - 48y + 44 = 0.$$

As this is the equation satisfied by the co-ordinates of any position of  $P$ , it represents the required equation to the locus of  $P$ .

The equation may be put in the form

$$x^2 + y^2 - 7\frac{1}{2}x - 4\frac{1}{2}y + 4\frac{1}{2} = 0,$$

$$\text{or, } (x - 3\frac{1}{2})^2 + (y - 4\frac{1}{2})^2 = (3\frac{1}{2})^2 + (4\frac{1}{2})^2 - 4\frac{1}{2} = 72,$$

$$\text{or, } \sqrt{(x - 7\frac{1}{2})^2 + (y - 4\frac{1}{2})^2} = 6\sqrt{2}.$$

This shows that the distance of the moving point  $x, y$  from the fixed point  $(7\frac{1}{2}, 4\frac{1}{2})$  is constant  $= 6\sqrt{2}$ . This identifies the locus to be a circle with centre  $(7\frac{1}{2}, 4\frac{1}{2})$  and radius  $6\sqrt{2}$ .

**Ex. 3.** *Prove that the equation*

$$9x^2 + 25y^2 - 108x + 100y + 199 = 0$$



by properly transferring the origin without turning the axes can be reduced to the form

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

The given equation can be written as

$$9(x^2 - 12x) + 25(y^2 - 4y) = -199,$$

$$\text{or, } 9(x-6)^2 + 25(y+2)^2 = 9.36 + 25.4 - 199 = 225.$$

Now transferring the origin to  $(6, -2)$  without turning the axes i.e., replacing  $x$  by  $x+6$  and  $y$  by  $y-2$ , [ See § 2.3 ] the new equation becomes

$$9x^2 + 25y^2 = 225, \text{ or, } \frac{x^2}{25} + \frac{y^2}{9} = 1.$$

### Examples II

1. A point moves so that (i) its distance from the  $y$ -axis is always  $-3$ , (ii) the sum of its distances from the axes of co-ordinates is always  $9$ . Write down the equation to its locus in each case.

2. Find the equation to the locus of a point which moves so that

(i) its distance from the point  $(-3, 7)$  is always equal to  $4$ .

(ii) its distance from the  $y$ -axis is always half its distance from the origin.

(iii) the rectangle contained by its distances from the axes is equal to a square of side  $c$ .

3. If the difference of the squares of the distances of a moving point from two fixed points is constant, show that it traces a straight line.

4.  $A$  and  $B$  are two fixed points whose co-ordinates are  $(-3, 4)$  and  $(5, -2)$  respectively. A point  $P$  moves in such a manner that the area of the triangle  $PAB$  remains constant. Show analytically that its locus is a straight line.

5. If  $A, B, C$  are three fixed points whose co-ordinates are  $(a, 0), (-a, 0), (c, 0)$  respectively, and if  $P$  be a movable point such that  $PA^2 + PB^2 = 2PC^2$ , find the equation to the locus of  $P$ .

6. The co-ordinates of three fixed points  $A, B, C$  are  $(3, 4)$ ,  $(-2, 5)$  and  $(-1, -9)$  respectively.  $P$  is a variable point such that  $PA^2 + PB^2 + PC^2 = \text{constant}$ . Prove that  $P$  traces out a circle.

7.  $A$  and  $B$  are two fixed points and  $P$  is a movable point such that  $PA : PB$  is constant. Show that the locus of  $P$  is a circle.

[ Choose the middle point of  $AB$  as origin, and  $AB$  as  $x$ -axis. ]

8. A point moves so that its distance from the fixed point  $(2a, 0)$  is equal to its distance from the  $y$ -axis. Determine the equation to its locus.

If the origin be transferred to  $(a, 0)$ , the axes remaining fixed in direction, prove that the transformed equation is  $y^2 = 4ax$ .

9. The sum of the distances of a movable point  $P$  from the two fixed points  $(c, 0)$  and  $(-c, 0)$  is a constant  $= 2a$ . Prove that the equation to the locus of  $P$  is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 - b^2 = c^2.$$

[ Let  $P(x, y)$ .  $\therefore \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$ ,

$$\text{or, } \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2},$$

$$\text{or, } a^2 - cx = a\sqrt{(x-c)^2 + y^2}.$$

Squaring,  $x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$ ,

$$\text{or, } x^2b^2 + a^2y^2 = a^2b^2,$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ which is the locus of } P.]$$

10. The distance between two fixed points  $S$  and  $S'$  is  $2c$ . Taking the middle point of  $SS'$  as origin, and  $SS'$  as  $x$ -axis, determine the equation to the locus of a point  $P$  which moves so that

$$PS - PS' = 2a \text{ (a constant).}$$

[ If the mid-point of  $SS'$  is taken as origin, the co-ordinates of  $S(c, 0)$  and  $S'(-c, 0)$  and let  $P(x, y)$ .

$$PS - PS' = 2a \text{ i.e., } \sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a,$$

$$\text{or, } \sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}.$$

Proceed as in Ex. 9. ]

11. Transform to parallel axes through the point  $(1, -2)$  the equations

$$(i) \ y^2 - 4x + 8 = 0. \quad (iii) \ 2x^2 + y^2 - 4x + 4y = 0.$$

12. What does the equation

$$(a-b)(x^2 + y^2) - 2abx = 0,$$

become if the origin be transferred to  $\frac{ab}{a-b}, 0$ , axes remaining unchanged in direction?

13. The equation to a curve being

$$(1-e^2)x^2 + y^2 + d^2 = 2dx,$$

find its equation if the origin be shifted to  $\frac{d}{1-e^2}, 0$ , without rotation of axes.

[Replacing  $x, y$  by  $x + \frac{d}{1-e^2}, y$  equation becomes  $(1-e^2)\left[\left(x + \frac{d}{1-e^2}\right)^2\right] + y^2 + d^2 = 2d\left(x + \frac{d}{1-e^2}\right),$

$$\text{or, } (1-e^2)x^2 + y^2 = \frac{d^2e^2}{1-e^2}.]$$

14. By transforming to parallel axes through a properly chosen point, prove that the equation

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$$

can be reduced to a homogeneous equation of the second degree in  $x, y$ .

#### ANSWERS

$$1. (i) \ x+3=0. \quad (ii) \ x+y=9. \quad 2. (i) \ x^2+y^2+6x-14y+42=0.$$

$$(ii) \ y=\pm x\sqrt{3}. \quad (iii) \ xy=c^2. \quad 5. \ x=\frac{c^2-a^2}{2c}.$$

$$8. \ y^2=4a(x-a). \quad 10. \ \frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1. \quad 11. (i) \ y^2=4x.$$

$$(ii) \ \frac{x^2}{9}+\frac{y^2}{6}=1. \quad 12. \ x^2+y^2=\left(\frac{ab}{a-b}\right)^2. \quad 13. \ x^2+\frac{y^2}{1-e^2}=\left(\frac{de}{1-e^2}\right)^2.$$

## CHAPTER III

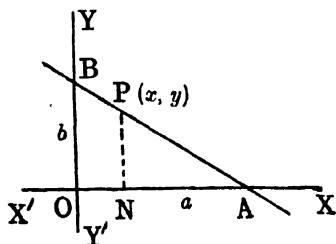
### STRAIGHT LINE

#### 3'1. Standard forms of the equation.

In the previous chapter we have seen that a first degree equation always represents a straight line. Now the position of a given straight line on a plane may be fixed up in various ways, and according to each way of defining the position of a line, we get a definite form of the equation to the line. The standard forms of the equation to a line are given below :

(A) *Intercepts on the axes are given in magnitude and sign.*

Let a straight line  $AB$  intersect the axes of  $x$  and  $y$  at  $A$  and  $B$  respectively, and let the intercepts  $OA$  and  $OB$  on the axes be  $a$  and  $b$  (given in magnitude and sign) so that the position of the line is definite.



Let  $(x, y)$  denote the co-ordinates of any point  $P$  on the line. Let  $PN$  be the perpendicular from  $P$  on  $OX$ , so that  $ON = x$ ,  $NP = y$ .

Now the triangles  $PNA$  and  $BOA$  are easily seen to be similar.

$$\therefore \frac{NP}{OB} = \frac{NA}{OA}, \text{ or } \frac{y}{b} = \frac{a-x}{a} = 1 - \frac{x}{a}$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1$$

which being the relation satisfied by the co-ordinates  $(x, y)$  of any point on the line, is the equation to the line.

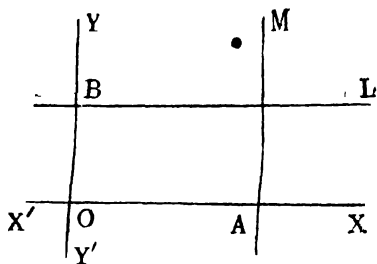
*Alternatively*

The co-ordinates of  $A$  and  $B$  are evidently  $(a, 0)$  and  $(0, b)$  respectively. Now  $P$  being on the straight line  $AB$ , the triangle  $PAB$  is of zero area.

Hence,  $\frac{1}{2}\{x(0-b) + a(b-y) + 0(y-0)\} = 0$

or,  $-xb + ab - ay = 0$ , whence  $\frac{x}{a} + \frac{y}{b} = 1$ .

(B) *Straight lines parallel to  $x$ -axis or  $y$ -axis.*



For a straight line  $BL$  parallel to the  $x$ -axis, if its intercept  $OB = b$ , then clearly the distance of every point of it from the  $x$ -axis being the same, namely  $b$ , the  $y$ -co-ordinate of every point of it is  $b$ , whatever its  $x$ -co-ordinate may be. Hence, its equation

$$y = b.$$

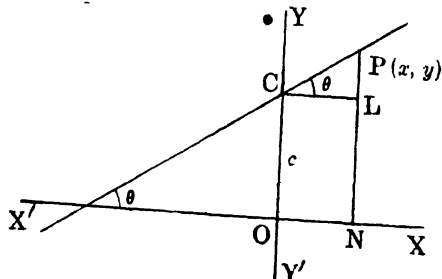
Similarly, for a straight line  $AM$  parallel to the  $y$ -axis, if its intercept  $OA$  on the  $x$ -axis be  $a$ , the equation will be

$$x = a.$$

Conversely, equations of the form  $x = a$ , or  $y = b$ , in which one co-ordinate is absent, represent straight lines parallel to the  $y$ -axis or to the  $x$ -axis, as the case may be.

**Note.** It may be noted that when a straight line becomes parallel to the  $y$ -axis, its intercept on the  $y$ -axis,  $b \rightarrow \infty$ , and so the equation to the line, from (A), reduce to the form  $\frac{x}{a} = 1$ , i.e.,  $x = a$ . Similarly, for lines parallel to the  $x$ -axis, the equation reduces to the form  $y = b$ .

(C) *Intercept on the  $y$ -axis, and inclination to the  $x$ -axis are given.*



Let a straight line, inclined at a given angle  $\theta$  to the  $x$ -axis, intersect to the  $y$ -axis at  $C$ , and let the intercept  $OC$  be given to

be  $c$  (in magnitude and sign).  $\theta$  and  $c$  being given, the position of the line is definitely fixed.

Let  $(x, y)$  be the co-ordinates of any point  $P$  on the line.  $PN$  being perpendicular from  $P$  on  $OX$ , we have  $ON = x$ ,  $NP = y$ . Now  $CL$  being perpendicular from  $C$  on  $NP$ ,  $LP = NP - NL = NP - OC = y - c$ , and  $CL = ON = x$ . Also  $\angle PCL = \theta$  ( $\neq \frac{1}{2}\pi$ ).

$$\tan \theta = \frac{LP}{CL} = \frac{y - c}{x}. \quad y = x \tan \theta + c.$$

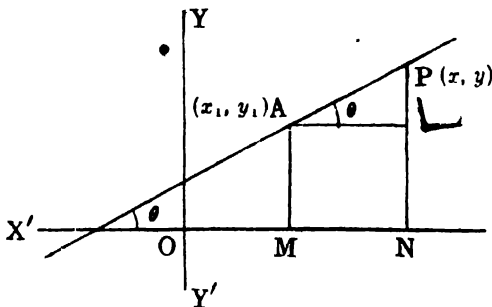
Denoting  $\tan \theta$  by  $m$ ,

$$y = mx + c \quad (m = \tan \theta) \quad (\theta \neq \frac{1}{2}\pi).$$

This being the relation satisfied by the co-ordinates of any point on the line, it represents the equation to the line.

**Note.**  $m (= \tan \theta)$  is defined as the *slope* or *gradient* of the line. The angle  $\theta$  may be acute or obtuse, and hence  $m$  may be positive or negative according to the position of the line. When the line is parallel to the  $x$ -axis,  $\theta = 0$ , and so  $m = 0$ , the equation to the line then reduces to the form  $y = c$  [ See (B) above ]. If  $\theta = 90^\circ$ ,  $m \rightarrow \infty$  or  $\frac{1}{0}$ , the equation to the line reduces to the form  $x = a$  in this case [ See (B) above ].

(D) *Straight line through a given point  $(x_1, y_1)$ , and making a given angle  $\theta$  with the  $x$ -axis.*



Let  $A$  be the given point  $(x_1, y_1)$  and  $P$  any point  $(x, y)$  on the line, whose inclination to the  $x$ -axis is  $\theta$ .

Let  $AM$ ,  $PN$  be perpendiculars on  $OX$ , and  $AL$  perpendicular on  $PN$ . Then,  $\angle PAL = \theta$ .

$$\text{Now, } \tan \theta = \frac{LP}{AL} = \frac{NP - MA}{ON - OM} = \frac{y - y_1}{x - x_1}.$$

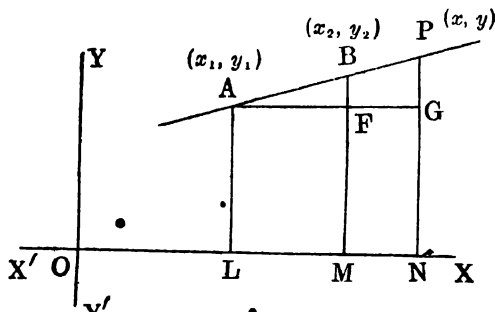
$$\therefore y - y_1 = (x - x_1) \tan \theta = m(x - x_1), \text{ where } m = \tan \theta,$$

$$\text{or, } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta},$$

any one of which represents the required equation to the line.

(E) *Straight line passing through two given points  $(x_1, y_1)$  and  $(x_2, y_2)$ .*

Let  $A(x_1, y_1)$  and  $B(x_2, y_2)$  be two given points through which a straight line passes, and  $P(x, y)$  be any point on it.



Draw  $AL$ ,  $BM$ ,  $PN$  perpendiculars on  $OX$ , and let  $AFG$  be drawn parallel to  $OX$ , intersecting  $BM$  and  $PN$  at  $F$  and  $G$  respectively.

Then, triangles  $PAG$  and  $BAF$  are evidently similar.

$$\frac{AG}{AF} = \frac{GP}{FB}, \text{ or, } \frac{ON - OL}{OM - OL} = \frac{NP - LA}{MB - LA}.$$

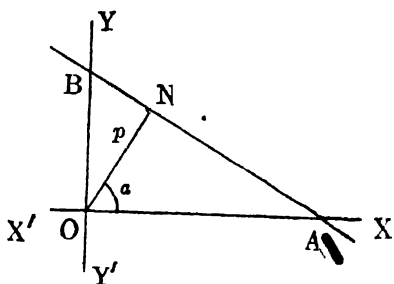
$$\therefore \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1},$$

$$\text{or, } y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1),$$

any one of which represents the required equation to the line.

The slope of the line,  $m = \frac{y_2 - y_1}{x_2 - x_1}$  in this case.

(F) Perpendicular distance  $p$  from the origin on the line, and the angle  $\alpha$  which this perpendicular makes with the  $x$ -axis are



Let the straight line intersect the axes at  $A$  and  $B$ , and let  $p$  be the perpendicular  $ON$  to it from the origin, and  $\alpha$  the angle  $XON$  made by  $ON$  with the  $x$ -axis.

Clearly,

$$OA = ON \sec \angle AON = p \sec \alpha,$$

$$\text{and } OB = ON \sec \angle NOB = p \sec (90^\circ - \alpha) = p \operatorname{cosec} \alpha.$$

These being the intercepts on the axes, the equation to the line is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1 \text{ or } x \cos \alpha + y \sin \alpha = p.$$

**Note.** Here  $p$  is always taken with a positive sign. The angle  $\alpha$  however may have any value, positive or negative, acute or obtuse. According to the quadrant in which  $ON$  lies, the signs of  $\cos \alpha$  and  $\sin \alpha$  will become definite.

### 3.2. Reduction of the most general equation $ax + by + c = 0$ of the first degree of a straight line to standard forms.

It may be noted that the most general form of the equation of a straight line, namely  $ax + by + c = 0$  can be reduced to any of the standard forms given above.

For instance, the above equation may be written as  $ax + by = -c$ ,

$$\text{or, } \frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1$$



which is of the form (A), showing that the intercepts on the axes of  $x$  and  $y$  are  $-\frac{c}{a}$  and  $-\frac{c}{b}$  respectively.

Again, we can write the above equation as  $by = -ax - c$ ,

$$\text{or, } y = -\frac{a}{b}x - \frac{c}{b}$$

which is of the form (C), showing that its slope on the  $x$ -axis,

or,  $m = -\frac{a}{b}$ , and its intercept on the  $y$ -axis is  $-\frac{c}{b}$ .

Further, the general equation  $ax + by + c = 0$  can be written as  $ax + by = -c$ , or dividing by  $\pm \sqrt{a^2 + b^2}$ , we can write it as

$$\pm \frac{a}{\sqrt{a^2 + b^2}}x + \pm \frac{b}{\sqrt{a^2 + b^2}}y = \pm \frac{-c}{\sqrt{a^2 + b^2}}$$

which is of the form (F), as can be evidenced by writing

$\cos \alpha = \pm \frac{a}{\sqrt{a^2 + b^2}}$ , whence  $\sin \alpha = \pm \frac{b}{\sqrt{a^2 + b^2}}$ . The perpendicular

from the origin on the line,  $p = \pm \frac{-c}{\sqrt{a^2 + b^2}}$  (where,

that sign in the denominator is to be taken which makes the perpendicular  $p$  positive in sign). The sign of the denominator

being thus fixed, the angle  $\alpha$  is given by  $\cos \alpha = \pm \frac{a}{\sqrt{a^2 + b^2}}$  and

$\sin \alpha = \pm \frac{b}{\sqrt{a^2 + b^2}}$ , which will have their magnitudes and signs

definite, and accordingly  $\alpha$  will be definitely known, i.e., the quadrant in which the perpendicular from the origin on the line falls is known and  $p$  being known, the position of the line is fixed up.

**Note.** That the intercepts of the line  $ax + by + c = 0$  on the axes are  $-c/a$  and  $-c/b$  can also be obtained by putting  $y = 0$  and  $x = 0$  respectively in the equation.

**3.3. Point of intersection of two given lines  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ .**

At the intersection point, both the equations being satisfied, we get by solving the equations as simultaneous equations,

$\frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} - \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$  giving the co-ordinates of the point of intersection.

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad \left[ \frac{a}{a_2} \neq \frac{c_1}{b_2} \right]$$

**3'4. Equation of a straight line through the point of intersection of two given lines  $a_1x + b_1y + c_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$ .**

(i) The co-ordinates of the point of intersection of the two given lines, by solving the simultaneous equations (as in § 3'3) are

$$x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, y_1 = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Now any line through  $x_1, y_1$  is  $y - y_1 = m(x - x_1)$  where  $m$  is arbitrary and different for different lines.

(ii) The equation  $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$   $k \neq 0$  is clearly a first degree equation, and thus represents a straight line. Again, at the intersection point of the two given lines the above equation is satisfied for all values of  $k$ , since each of the bracketed portions is zero there. Thus the equation always represents a straight line passing through the intersection of the two given lines. For different values of  $k$ , it represents different such lines.

**3'5. Condition that three given lines,  $a_1x + b_1y + c_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$ ,  $a_3x + b_3y + c_3 = 0$  may be concurrent.**

The point of intersection of the last two lines (by solving them as simultaneous equations) is easily seen to have co ordinates

$$x = \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, y = \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}$$

The first line will pass through this point only if the above co-ordinates satisfy its equation. Hence, the required condition that the three lines may be concurrent is

$$a_1 \left( \frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} \right) + b_1 \left( \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2} \right) + c_1 = 0$$

or,  $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$ . (A

*Alternative condition :*

If any three non-zero constants  $l, m, n$  can be found such that

$$l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) + n(a_3x + b_3y + c_3) = 0. \quad (B)$$

*Identically*, then the above three lines are concurrent.

If  $(x', y')$  be the point of intersection of the first two lines, then  $a_1x' + b_1y' + c_1 = 0$  and  $a_2x' + b_2y' + c_2 = 0$  and hence substituting  $x', y'$  for  $x, y$ , in the given relation (B), we get

$$n(a_3x' + b_3y' + c_3) = 0, \text{ i.e., } a_3x' + b_3y' + c_3 = 0$$

i.e., the third line  $a_3x + b_3y + c_3 = 0$ , passes through  $(x', y')$ , the point of intersection of the first and second i.e., the three lines are concurrent.

**3'6. Condition that three given points  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  are collinear.**

$A, B, C$  denoting the given points, if they be collinear, the triangle formed by them is of zero area. For this the condition is [ from § 1'4 ]

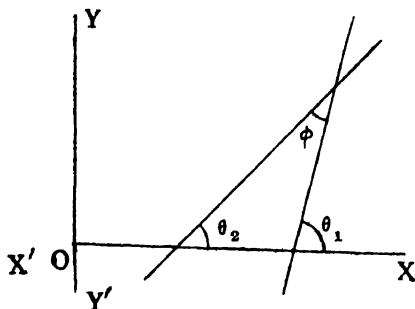
$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

**Note.** The above condition can also be deduced by writing down the equation of the straight line through any two of the points, and using the condition that this equation is satisfied by the co-ordinates of the third point.

**3'7. Angle between two given lines.**

(A) *When the lines are given by the equations*

$$y = m_1x + c_1, \quad y = m_2x + c_2.$$



Let  $\theta_1$  and  $\theta_2$  be the angles made by the given lines with the  $x$ -axis.

Then,

$$\tan \theta_1 = m_1, \quad \tan \theta_2 = m_2$$

Now  $\phi$  being the angle of intersection between the two lines, clearly

$$\phi = \theta_1 - \theta_2.$$

$$\therefore \tan \phi = \frac{\tan \theta_1 \sim \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} = \frac{m_1 \sim m_2}{1 + m_1 m_2}.$$

Thus,  $\phi$  is known.

(B) When the lines are given by the equations

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Here the slopes of the lines are clearly given by

$$m_1 = -\frac{a_1}{b_1}, \quad m_2 = -\frac{a_2}{b_2}.$$

Hence, by (A) above,

$$\tan \phi = \frac{\left(-\frac{a_1}{b_1}\right) \sim \left(-\frac{a_2}{b_2}\right)}{1 + \frac{a_1 a_2}{b_1 b_2}} = \frac{a_1 b_2 \sim a_2 b_1}{a_1 a_2 + b_1 b_2}.$$

Cor. The two lines will be *parallel* to one another when  $\phi = 0$ ; hence the condition of parallelism is

$$m_1 = m_2 \text{ or } \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

The two lines will be *perpendicular* to one another when  $\phi = \frac{1}{2}\pi$  or  $\cot \phi = 0$ , giving

$$1 + m_1 m_2 = 0, \text{ or } m_1 m_2 = -1, \text{ or } a_1 a_2 + b_1 b_2 = 0$$

as the necessary condition-

Note. From above, it is clear that any straight line parallel to  $y = m_1x + c_1$  may have its equation taken as

$$y = m_1x + c_2, \text{ where } c_2 \text{ is arbitrary.}$$

Similarly, any line parallel to  $a_1x + b_1y + c_1 = 0$  may have its equation assumed in the form  $a_1x + b_1y + c_2 = 0$ , where  $c_2$  is different for different such lines.

Again, any line perpendicular to  $y = m_1x + c_1$  can be taken to be  $y = -\frac{1}{m_1}x + c_2$ , and any line perpendicular to  $a_1x + b_1y + c_1 = 0$  may be assumed as  $b_1x - a_1y + c_2 = 0$ , where  $c_2$  is arbitrary in both cases.

These assumptions are very useful in working out examples.

## 3.8. Illustrative Examples.

**Ex. 1.** Find the equation of the straight line passing through the point  $(-5, 3)$  and perpendicular to the line  $x - 2y + 6 = 0$ .

Any line through  $(-5, 3)$  has its equation of the form [ See § 3.1, D ]

$$y - 3 = m(x + 5) \quad \dots \quad (i)$$

The given line is  $2y = x + 6$  or  $y = \frac{1}{2}x + 3$ , showing that its slope 'm' is  $\frac{1}{2}$ .

Now, (i) being perpendicular to this line, the necessary condition gives

$$m \times \frac{1}{2} = -1 \text{ or } m = -2.$$

Hence, (i) becomes

$$y - 3 = -2(x + 5), \text{ or, } 2x + y + 7 = 0$$

which is the equation of the required line.

**Ex. 2.** Obtain the equation of the straight line which passes through the intersection of the lines  $2x - 3y + 4 = 0$  and  $3x + 4y - 5 = 0$ , and has equal intercepts of the same sign along the axes. Find the perpendicular distance from the origin on this straight line.

Any line passing through the intersection of the two given lines can have its equation written in the form

$$2x - 3y + 4 + k(3x + 4y - 5) = 0,$$

$$\text{or, } (2 + 3k)x + (4k - 3)y + (4 - 5k) = 0 \quad \dots (i)$$

Its intercept on the x-axis (by putting  $y = 0$  in the above equation) is  $\frac{5k - 4}{2 + 3k}$ .

Similarly, (putting  $x = 0$ ) the intercept on the y-axis is  $\frac{5k - 4}{4k - 3}$ .

As the two intercepts are equal in magnitude, and are of the same sign, we have

$$\frac{5k - 4}{2 + 3k} = \frac{5k - 4}{4k - 3}, \text{ or, } 2 + 3k = 4k - 3, \text{ whence } k = 5.$$

Substituting in (i), the equation of the required straight line is

$$17x + 17y - 21 = 0, \text{ or, } 17x + 17y = 21 \quad \dots (ii)$$

If  $p$  denotes the perpendicular distance on it from the origin, and if  $a$  be

the angle which this perpendicular makes with the positive direction of the  $x$ -axis, then the equation of the line can be written as

$$x \cos \alpha + y \sin \alpha = p \quad \quad \quad [\text{See § 3'1 (F)}]$$

As this is identical with equation (ii), comparing coefficients, we get

$$\frac{\cos \alpha}{17} = \frac{\sin \alpha}{21} = \frac{p}{21}, \quad \cos \alpha = \frac{17p}{21}, \quad \sin \alpha = \frac{17p}{21}.$$

$$\therefore \left(\frac{17p}{21}\right)^2 + \left(\frac{17p}{21}\right)^2 = 1 \text{ giving } p^2 = \frac{21^2}{17^2 + 17^2}$$

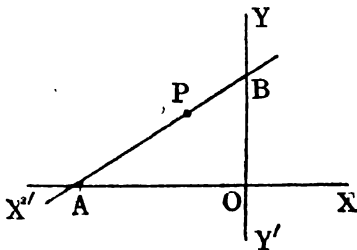
$$\text{whence, } p = \frac{21}{17\sqrt{2}}.$$

As the perpendicular from the origin on a line is always taken as positive, the positive sign of the square root is taken.

**Ex. 3.** Find the equation to the straight line which passes through the point  $(-4, 3)$  and is such that the portion of the line between the axes is divided at the point internally in the ratio  $5 : 3$ .

Let  $P$  be the point  $(-4, 3)$ , and let  $APB$  be the line intersecting the axes at  $A$  and  $B$ , such that  $AP : PB = 5 : 3$ .

Let the intercepts  $OA$  and  $OB$  on the axes be  $a$  and  $b$  respectively in magnitude and sign. Then co-ordinates of  $A$  are evidently  $(a, 0)$  and those of  $B$  are  $(0, b)$ .



The co-ordinates of the point  $P$  which divides  $AB$  in the ratio  $5 : 3$  are then

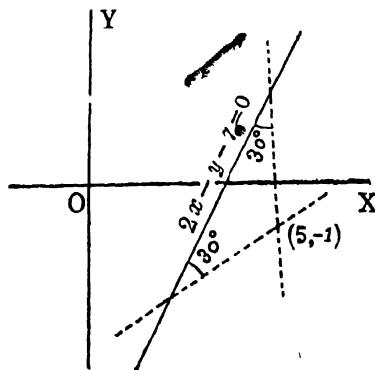
$$\frac{5 \cdot 0 + 3 \cdot a}{5 + 3} \text{ and } \frac{5 \cdot b + 3 \cdot 0}{5 + 3} \text{ i.e., } \frac{3a}{8} \text{ and } \frac{5b}{8}.$$

Thus,  $\frac{3a}{8} = -4$  and  $\frac{5b}{8} = 3$ . Hence,  $a = -\frac{32}{3}$ ,  $b = \frac{24}{5}$ . The equation to

the line  $AB$  then, having intercepts  $-\frac{32}{3}$  and  $\frac{24}{5}$  along the axes is

$$\frac{3x}{-32} + \frac{5y}{24} = 1, \text{ i.e., } 9x - 20y + 96 = 0.$$

**Ex. 4.** Find the equations to the straight lines passing through the point  $(5, -1)$  and making an angle of  $30^\circ$  with the line  $2x - y - 7 = 0$ .



Any line passing through  $(5, -1)$  is  $y + 1 = m(x - 5) \dots (i)$ . The given line

$$y = 2x - 7$$

has its ' $m$ ' = 2.

If (i) makes an angle  $30^\circ$  with it, we have

$$\tan 30^\circ = \frac{2 - m}{1 + 2m}.$$

Thus,  $\frac{1}{\sqrt{3}} = \frac{2 - m}{1 + 2m}$ , or  $\frac{m - 2}{1 + 2m}$ .

$$= \frac{2\sqrt{3} - 1}{2 + \sqrt{3}}, \text{ or } -\frac{2\sqrt{3} + 1}{2 - \sqrt{3}}.$$

Hence, the equation to the required lines are

$$y + 1 = \frac{2\sqrt{3} - 1}{2 + \sqrt{3}}(x - 5), \text{ and } y + 1 = -\frac{2\sqrt{3} + 1}{2 - \sqrt{3}}(x - 5).$$

### Examples III(a)

1. Obtain the equation to the line inclined to the  $x$ -axis at an angle  $60^\circ$ , and having its intercept on the  $x$ -axis equal to  $-1$ .
2. Find the equation to the line passing through the point  $(-7, 3)$  and having its intercept on the  $x$ -axis equal to  $-5$ .
3. Find the equation to a straight line passing through the point  $(-2, -3)$ , and having equal intercepts of opposite sign on the axes.
4. Find the equations to the straight lines joining the pair of points
  - (i)  $(-3, 7)$  and  $(-1, -2)$ ;
  - (ii)  $(0, -3)$  and  $(7, 0)$ ;
  - (iii)  $(-2, 6)$  and  $(-2, -4)$ .
5. Show that the equations of the straight lines joining the points

(i)  $(at_1^2, 2at_1)$  and  $(at_2^2, 2at_2)$  is

$$y(t_1 + t_2) - 2x = 2at_1t_2.$$

(ii)  $(a \cos \theta, b \sin \theta)$  and  $(a \cos \theta', b \sin \theta')$  is

$$\frac{x}{a} \cos \frac{1}{2}(\theta + \theta') + \frac{y}{b} \sin \frac{1}{2}(\theta + \theta') = \cos \frac{1}{2}(\theta - \theta').$$

6. Find the equation to the straight line joining the origin to the point of intersection of the lines  $2x - 5y = 1$  and  $x + 3y + 7 = 0$ .

7. Find the equation to the straight line passing through the point  $(-2, 1)$  and also through the intersection of the lines  $2x - 3y + 5 = 0$  and  $x + 4y - 7 = 0$ .

8. (i) Obtain the equation to the straight line passing through the point  $(-1, 2)$  and perpendicular to the line  $3x + 4y = 5$ .

[ H. S. 1960, Compartmental ]

[ Equation of line through

$$(-1, 2) \text{ is } y - 2 = m(x + 1).$$

Since this line  $\perp$  to

$$3x + 4y = 5, \text{ hence } mx - \frac{1}{4} = -1. \quad [ \S 3.7 ]$$

Putting the value of  $m$  and simplifying, equation of line is obtained.

$$\therefore m = \frac{3}{4}.$$

(ii) Find the co-ordinates of the foot of the perpendicular from the point  $(8, -6)$  on the straight line  $2x - 3y + 5 = 0$ .

9. (i) Show that the points  $(1, 4)$ ,  $(3, -2)$  and  $(-3, 16)$  are collinear.

(ii) Prove that the points whose co-ordinates are respectively  $(5, 1)$ ,  $(1, -1)$  and  $(11, 4)$  lie on a straight line and find the intercepts of this line on the axes. [ H. S. 1961, Comp. ]

[ Equation of this line through

$$(5, 1) \text{ and } (1, -1) \text{ is } \frac{x-5}{1-5} = \frac{y-1}{-1-1} \text{ or } x - 2y - 3 = 0$$

which is satisfied by  $(11, 4)$ . Hence the points are collinear. ]

10. If the points  $(a, b)$ ,  $(a', b')$ ,  $(a - a', b - b')$  are collinear, show that their join passes through the origin, and  $ab' = a'b$ .



11. (i) Prove that the three lines  $2x - 7y + 10 = 0$ ,  $3x - 2y - 2 = 0$  and  $x - 7y + 12 = 0$  meet in a point.

(ii) Find for what value of  $k$  the three lines  $x - y + 5 = 0$ ,  $x + y = 1$  and  $y = kx + 13$  will be concurrent.

Find also the point of concurrence.

✓ 12. Find the equation to the perpendicular bisector of the straight line joining the points  $(-2, 7)$  and  $(8, -1)$ . At what distance is this perpendicular bisector from the origin?

[ H. S. 1961 ]

13. Determine the equation of the straight line which passes through the intersection of the lines given by  $3x - 4y + 1 = 0$  and  $5x + y = 1$ , and has equal intercepts of the same sign along the axes.

[ H. S. 1960 ]

14. A straight line passes through the intersection of the lines  $3x - 7y + 5 = 0$ ,  $x - 2y - 7 = 0$ , and has equal intercepts of the same sign along the axes. Find the length of the perpendicular on it from the origin.

[ H. S. 1961, *Compartmental* ]

15. A straight line is drawn through the point  $(3, 5)$  such that the point bisects the portion of the line intercepted between the axes. Find the equation to the line, and calculate its perpendicular distance from the origin.

[ H. S. 1960, *Comp.* ]

—16. The portion of a straight line intercepted between the axes is divided internally in the ratio  $2 : 3$  at the point  $(-9, -2)$ . Show that it passes through the point  $(0, -5)$  and is perpendicular to the line  $y = 3x$ .

17. (i) Find the equation of the line joining the point of intersection of the lines  $x + 3y + 2 = 0$ ,  $2x - y - 3 = 0$  to the point of intersection of  $7x - y - 3 = 0$ ,  $2x - 5y - 15 = 0$ .

(ii) Find the equations of the diagonals of the parallelogram formed by the lines  $4x - 5y - 7 = 0$ ,  $4x - 5y - 14 = 0$ ,  $3x + 7y - 8 = 0$ ,  $3x + 7y - 12 = 0$ .

18. Determine the angle between the lines  $y = \frac{1}{2}(x - 5)$  and  $+3y = 2$ . Also find out the equation to the line through their point of intersection making a positive angle  $60^\circ$  with the positive direction of the  $x$ -axis.

## STRAIGHT LINES

19. Show that the lines  $x \cos \alpha + y \sin \alpha = p$ ,

$$x \cos \left( \alpha + \frac{2}{3}\pi \right) + y \sin \left( \alpha + \frac{2}{3}\pi \right) = p \text{ and}$$

$x \cos \left( \alpha - \frac{2}{3}\pi \right) + y \sin \left( \alpha - \frac{2}{3}\pi \right) = p$  form an equilateral triangle.

[ *Angles between each pair is  $60^\circ$*  ]

[ As in § 8·7(A), from the first two lines

$$m_1 = -\cot \alpha \text{ and } m_2 = -\cot \left( \alpha + \frac{2}{3}\pi \right).$$

$$\begin{aligned} \therefore \tan \theta &= \frac{-\cot \alpha + \cot \left( \alpha + \frac{2}{3}\pi \right)}{1 + \cot \alpha \cot \left( \alpha + \frac{2}{3}\pi \right)} = \cot \left[ \alpha - \left( \alpha + \frac{2}{3}\pi \right) \right] = -\tan \left( \frac{2}{3}\pi \right) \\ &= -\tan (180^\circ - 60^\circ) = \tan 60^\circ. \quad \therefore \theta = 60^\circ. \end{aligned}$$

Again considering the last two lines, we have

$$\tan \theta = \frac{-\cot \left( \alpha + \frac{2}{3}\pi \right) + \cot \left( \alpha - \frac{2}{3}\pi \right)}{1 + \cot \left( \alpha + \frac{2}{3}\pi \right) \cot \left( \alpha - \frac{2}{3}\pi \right)}$$

$$\cot \left[ \left( \alpha + \frac{2}{3}\pi \right) - \left( \alpha - \frac{2}{3}\pi \right) \right]$$

$$= \frac{1}{\cot \frac{4}{3}\pi} = \tan \left( \pi + \frac{1}{3}\pi \right) = \tan \frac{4}{3}\pi$$

$$= \tan 60^\circ$$

$$\therefore \theta = 60^\circ$$

Thus each of the two angles of a triangle being  $60^\circ$ , the third angle must be  $60^\circ$ . Hence the triangle is equilateral.

20. Find the equations of the two lines through the point  $(3, -1)$  which make an angle  $45^\circ$  with the line  $2x - y = 2$ .

21. In any triangle, show analytically that

(i) the medians are concurrent

[ We take  $BC$  as  $x$ -axis and the mid-point of  $BC$ , say  $D$ , as origin. Let the co-ordinates of  $B, C$  be  $(-a, 0)$  and  $(a, 0)$ . Equation of  $AD$  is

$$\frac{x-x_1}{0-x_1} = \frac{y-y_1}{0-y_1}, \text{ or, } xy_1 - x_1y = x_1y_1 - x_1y_1 = 0. \quad \dots (1)$$

Mid-point of  $AC$ , say  $E$ , has co-ordinates  $\left(\frac{x_1+a}{2}, \frac{y_1}{2}\right)$ .

Equation of  $BE$  is  $\frac{x+a}{\frac{x_1+a}{2}-a} = \frac{y-0}{\frac{y_1}{2}-0}$ ,

$$\text{or, } \frac{x+a}{x_1+a-a} = \frac{y}{\frac{y_1}{2}-0} \quad \text{or, } xy_1 - y(x_1+3a) + ay_1 = 0. \quad \dots (2)$$

Mid-point of  $AB$  is  $\left(\frac{x_1-a}{2}, \frac{y_1}{2}\right)$ .

Equation of  $CF$  is  $\frac{x-a}{\frac{x_1-a}{2}-a} = \frac{y-0}{\frac{y_1}{2}-0}$

$$\text{or, } xy_1 - y(x_1-3a) - ay_1 = 0.$$

Solving the last two equations, the co-ordinates of the point of intersection of two medians are obtained and the co-ordinates are  $\left(\frac{1}{3}x_1, \frac{1}{3}y_1\right)$ . But  $\left(\frac{1}{3}x_1, \frac{1}{3}y_1\right)$  is satisfied by the equation (1). Hence they are concurrent.]

(ii) the perpendicular bisectors of the sides are concurrent.

[ Taking the co-ordinates of vertices of  $\triangle ABC$  as in Ex. 21 (i), equation of line through  $D$  perpendicular to  $BC$  is  $x=0$ . ... (1)

Equation of line  $AB$  is  $\frac{x+a}{x_1+a} = \frac{y-0}{y_1-0}$ , or,  $xy_1 - y(x_1+a) + ay_1 = 0$ .

$$[ \text{'m' of this line} = \frac{y_1}{x_1+a} ]$$

Equation of line through  $F$  (mid-point of  $AB$ ) is  $y - \frac{y_1}{2} = m \left(x - \frac{x_1-a}{2}\right)$ .

If this line is perpendicular to  $AB$ ,  $m \cdot \frac{y_1}{x_1+a} = -1$ .  $\therefore m = -\frac{x_1+a}{y_1}$ .

Hence equation of line through  $F$  perpendicular to  $AB$  is

$$y - \frac{y_1}{2} = -\frac{x_1+a}{y_1} \left(x - \frac{x_1-a}{2}\right), \text{ or, } 2yy_1 + 2x(x_1+a) = y_1^2 + x_1^2 - a^2 \dots (2)$$

Similarly equation of line through  $E$  (mid-point of  $AC$ ) and perpendicular to  $AC$  is

$$y - \frac{y_1}{2} = -\frac{x_1-a}{y_1} \left(x - \frac{x_1+a}{2}\right) \text{ or } 2yy_1 + 2x(x_1+a) = x^2 + y_1^2 - a^2 \dots (3)$$

Solving (1) and (2), the point of intersection of the two bisectors is  $\left(0, \frac{x_1^2 + y_1^2 - a^2}{2y_1}\right)$  which is satisfied by equation (3). Hence perpendicular bisectors are concurrent.]

(iii) the perpendiculars from the vertices on opposite sides are concurrent.

[ Taking the co-ordinates of vertices of  $\triangle ABC$  in Ex. 21 (i). equation of line  $AB$  is  $xy_1 - y(x_1 + a) + ay_1 = 0$ .

$$\left[ \text{as in Ex. 21 (ii), 'm' of this line is } \frac{y_1}{x_1 + a} \right]$$

Hence equation of line through  $C$  perpendicular to  $AB$  is

$$y - 0 = m(x - a), \text{ or, } y = -\frac{x_1 + a}{y_1}(x - a) \left[ \text{since } m \frac{y_1}{x_1 + a} = -1 \right]$$

$$\text{or, } yy_1 + (x - a)(x_1 + a) = 0 \quad \dots (1)$$

Similarly equation of line through  $B$  perpendicular to  $AC$  is

$$yy_1 + (x + a)(x_1 + a) = 0 \quad \dots (2)$$

Equation of line through  $A$  perpendicular to  $BC$  is  $x = x_1 \quad \dots (3)$

Solving (1) and (3), the point of intersection is  $(x_1, -\frac{x_1^2 - a^2}{y_1})$  which is satisfied by (2). Hence the result.]

22. Find the ortho-centre of the triangle

(i) whose vertices are  $(2, 7)$ ,  $(-6, 1)$  and  $(4, -5)$ .

(ii) whose sides have equations  $2x + 7y + 24 = 0$ ,  $4x + y = 4$ , and  $x - 3y = 1$ .

23. Find the area of the triangle formed by the straight lines whose equations are  $x + 2y - 5 = 0$ ,  $y + 2x - 7 = 0$  and  $x - y + 1 = 0$ .

Find also the co-ordinates of the circum-centre of the triangle.

[ Solving the equations pair-wise the co ordinates of three vertices  $A, B, C$  of the triangle are  $(3, 1)$ ,  $(2, 3)$  and  $(1, 2)$ .

From § 1.4, area of  $\triangle ABC = \frac{1}{2}[3(3 - 2) + 2(2 - 1) + 1(1 - 3)] = \frac{3}{2}$ .

Let  $P(x, y)$  be the co-ordinates of circum-centre. Since circum-centre is equidistant from three vertices of  $\triangle ABC$ , hence

$$(x - 3)^2 + (y - 1)^2 = (x - 2)^2 + (y - 3)^2 = (x - 1)^2 + (y - 2)^2$$

which gives on simplification  $2x - 4y + 3 = 0$ ,  $x + y - 4 = 0$ . Solving, the co-ordinates of the circum-centre are  $(\frac{1}{2}, \frac{7}{2})$ .]

24. Show that the feet of the perpendiculars from the origin to the lines

$$x + y - 4 = 0, x + 5y - 26 = 0, 15x - 27y - 424 = 0$$

all lie on a straight line, and find the equation of this line.

[ 'm' of the line  $x + y - 4 = 0$  is  $-1$ .

Equation of line through (0, 0) perpendicular to  $x + y - 4 = 0$  is

$$y = mx = x \text{ [ since } mx - 1 = -1. \therefore m = 1 \text{ ] or, } y - x = 0.$$

Similarly equation of line through (0, 0) perpendicular to  $x + 5y - 6 = 0$  is  $y - 5x = 0$ .....(0, 0) perpendicular to  $15x - 27y - 424 = 0$  is  $27x + 15y = 0$ .

Solving  $y - x = 0$  and  $x + y - 4 = 0$ , the foot of the perpendicular is (2, 2).

Solving  $y - 5x = 0$  and  $x + 5y - 6 = 0$ , the foot of the perpendicular is (1, 5).

Solving  $27x + 15y = 0$  and  $15x + 27y - 424 = 0$ , the foot of the perpendicular is  $(-\frac{23}{3}, -12)$ .

Equation of line through (2, 2) and (1, 5) is  $\frac{x-2}{1-2} = \frac{y-2}{5-2}$ , or,  $3x + y = 8$  which is satisfied by  $(-\frac{23}{3}, -12)$  and hence lie on one straight line.]

25. A line moves so that the sum of the reciprocals of its intercepts on the two axes is constant. Show that it always passes through a fixed point.

[ Let the equation of line be  $\frac{x}{a} + \frac{y}{b} = 1$  (as in § 31 A)

$$\text{and } \frac{1}{a} + \frac{1}{b} = \text{const.} = k, \text{ say.}$$

$$\text{We can write } \frac{x}{a} + \frac{y}{b} = 1 = \frac{1}{k}; k = \frac{1}{k} \left( \frac{1}{a} + \frac{1}{b} \right)$$

$$\text{or, } \frac{1}{a} \left( x - \frac{1}{k} \right) + \frac{1}{b} \left( y - \frac{1}{k} \right) = 0. \text{ Therefore the line always passes through } (1/k, 1/k). ]$$

### ANSWERS

1.  $y = \sqrt{3}(x+1).$

2.  $3x + 2y + 15 = 0.$

3.  $x - y = 1$

4. (i)  $9x + 2y + 13 = 0.$

(ii)  $3x - 7y = 21.$

(iii)  $x = -2.$

6.  $32y = 15x.$

7.  $8x - 23y + 39 = 0.$

8. (i)  $4x - 3y + 10 = 0.$  (ii) 2, 3.

# STRAIGHT LINES

9. (ii) 3 and  $-1\frac{1}{2}$ .      11. (ii)  $k=5$ ;  $-2, 3$ .      12.  $5x-4y=3$ ;  $\sqrt{41}$ .

13.  $23x+23y=11$ .      14.  $85/\sqrt{2}$ .      15.  $\frac{x}{6} + \frac{y}{10} = 1$ ;  $\frac{1}{17}\sqrt{34}$ .

17. (i)  $2x-y=3$ ,      (ii)  $5x+69y=28$ ,  $37x+29y=112$ .

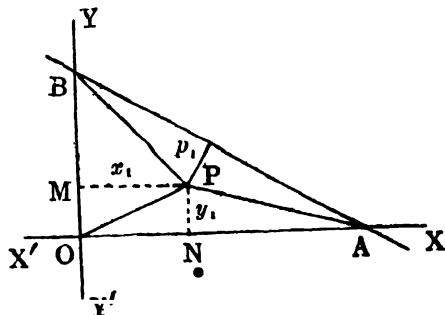
18.  $45^\circ$ ;  $5y+3=\sqrt{3}(5x-19)$ .      20.  $3x+y=8$ ,  $x-3y=6$ .

22.  $(-\frac{1}{3}, \frac{4}{3})$ ;  $(\frac{1}{3}, -\frac{7}{3})$ .      23.  $\frac{3}{2}$ ;  $(\frac{1}{8}, \frac{1}{8})$ .      24.  $3x+y=8$ .

**3.9. Length of perpendicular from a given point to a given straight line.**

(i) *Perpendicular from the point  $P(x_1, y_1)$  to the straight line  $AB$  given by the equation  $ax+by+c=0$ .*

$A$  and  $B$  being the points of intersection of the line with the axes, from its equation  $ax+by+c=0$ , the intercepts



$OA$  and  $OB$  (putting  $y=0$  and  $x=0$  respectively) are  $-\frac{c}{a}$  and

$$\therefore AB = \sqrt{\left(-\frac{c}{a}\right)^2 + \left(-\frac{c}{b}\right)^2} = \pm \frac{c}{ab} \sqrt{a^2 + b^2}.$$

$PN$  and  $PM$  being the perpendiculars from  $P(x_1, y_1)$  on the axes,  $PM=x_1$ ,  $PN=y_1$ .

Let  $p_1$  be the required perpendicular from  $P$  on  $AB$ .

*First method :*

$$\triangle PAB = \triangle OAB - \triangle POA - \triangle POB.$$

$$\therefore \frac{1}{2}p_1 \cdot AB = \frac{1}{2}OA \cdot OB - \frac{1}{2}PN \cdot OA - \frac{1}{2}PM \cdot OB,$$

$$\text{or, } p_1 \cdot \left( \pm \frac{c}{ab} \sqrt{a^2 + b^2} \right) = \left( -\frac{c}{a} \right) \left( -\frac{c}{b} \right) - y_1 \left( -\frac{c}{a} \right) - x_1 \left( -\frac{c}{b} \right),$$

$$\text{or, } \pm p_1 \sqrt{a^2 + b^2} = ax_1 + by_1 + c.$$

$$\therefore p_1 = \frac{ax_1 + by_1 + c}{\pm \sqrt{a^2 + b^2}}.$$

*Second method :*

$P, A, B$  have co-ordinates  $(x_1, y_1), \left( -\frac{c}{a}, 0 \right)$  and  $\left( 0, -\frac{c}{b} \right)$  respectively.

$$\therefore \Delta PAB = \frac{1}{2} \left\{ x_1 \left( 0 + \frac{c}{b} \right) + \left( -\frac{c}{a} \right) \left( -\frac{c}{b} - y_1 \right) + 0(y_1 - 0) \right\}.$$

$$\therefore \frac{1}{2} p_1 \cdot AB = \frac{1}{2} \left( \frac{cx_1}{b} + \frac{c^2}{ab} + \frac{cy_1}{a} \right),$$

$$\text{or, } p_1 \left( \pm \frac{c}{ab} \sqrt{a^2 + b^2} \right) = \frac{c(ax_1 + by_1 + c)}{ab}.$$

$$\therefore p_1 = \frac{ax_1 + by_1 + c}{\pm \sqrt{a^2 + b^2}}.$$

**Note.** To choose the proper sign of the denominator, it should be noted that the perpendicular from the origin  $(0, 0)$  to the line is always taken as positive.

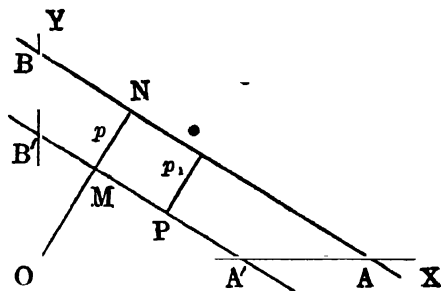
$\therefore$  the sign of the denominator should be taken same as that of  $c$ , as is apparent by putting  $x_1 = 0, y_1 = 0$ .

The perpendicular distance of the point  $P$  from the line  $AB$  is thus positive, if  $P$  lies on the same side as the origin with respect to the line  $AB$ . For this case  $ax_1 + by_1 + c$  has the same sign as that of  $c$ . If  $P$  be on the opposite side of the origin with respect to the line  $AB$ , the perpendicular from  $P$  on  $AB$  will be negative, and in this case,  $ax_1 + by_1 + c$  will have its sign opposite to that of  $c$ . In this connection, see § 3.10.

(ii) *Perpendicular from the point  $P(x_1, y_1)$  to the line  $x \cos \alpha + y \sin \alpha = p$ .*

Here, from the equation to the line, the perpendicular from the origin on it is  $ON = p$ , and  $ON$  makes an angle  $\alpha$  with the  $x$ -axis.

Let  $p_1$  be the perpendicular from the given point  $P(x_1, y_1)$  to the given line  $AB$ . If  $A'B'$  denotes the parallel line through  $P$ , clearly  $ON$  is perpendicular to  $A'B'$  as well, and the perpendicular  $OM$  from  $O$  on  $A'B'$  is clearly  $p - p_1$ .



Hence, equation to the line  $A'B'$  is

$$x \cos \alpha + y \sin \alpha = p - p_1.$$

As it passes through  $P(x_1, y_1)$ , we must have

$$x_1 \cos \alpha + y_1 \sin \alpha = p - p_1.$$

$$\therefore p_1 = p - x_1 \cos \alpha - y_1 \sin \alpha.$$

**Note.** Case (i) can be deduced as a Corollary to (ii), for the equation to the line being of the form  $ax + by + c = 0$ , it is reduced to the form of (ii) by dividing by  $\pm \sqrt{a^2 + b^2}$  [See § 3.9]. Thus  $\frac{ax + by + c}{\pm \sqrt{a^2 + b^2}}$  (when proper sign is selected) is of the form  $p - x \cos \alpha - y \sin \alpha$ ,  $p$  being positive and  $\pm \frac{c}{\sqrt{a^2 + b^2}}$  also positive by proper choice of sign.

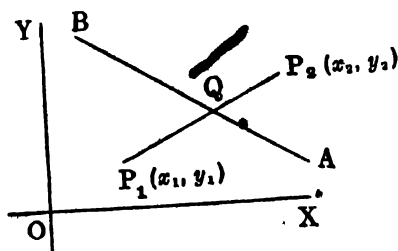
### 3.10. Theorem.

*The points  $(x_1, y_1)$  and  $(x_2, y_2)$  are on the same side or opposite sides of the straight line  $ax + by + c = 0$ , according as the expressions  $ax_1 + by_1 + c$  and  $ax_2 + by_2 + c$  are of the same or opposite signs.*



Let  $P_1$  and  $P_2$  be the points  $(x_1, y_1)$  and  $(x_2, y_2)$  and  $AB$  the straight line

$$ax + by + c = 0. \quad \dots (i)$$



Let the line  $P_1P_2$  intersect  $AB$  at  $Q$  where the ratio  $P_1Q : QP_2$  is  $\lambda : 1$ .

Then the co-ordinates of  $Q$  are

$$\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}.$$

As  $Q$  lies on  $AB$ , these co-ordinates must satisfy the equation (i).

$$\text{Hence, } a \frac{\lambda x_2 + x_1}{\lambda + 1} + b \frac{\lambda y_2 + y_1}{\lambda + 1} + c = 0,$$

$$\text{or, } \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0;$$

$$\therefore \lambda = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$$

If  $P_1$  and  $P_2$  be on opposite sides of  $AB$ ,  $Q$  must be within  $P_1P_2$  and so the ratio  $\lambda : 1$  is positive; if  $P_1$  and  $P_2$  be on the same side of  $AB$ ,  $Q$  must be external to  $AB$  and so  $\lambda : 1$  is negative.

Hence,  $P_1$  and  $P_2$  will be on the same or opposite sides of  $AB$  according as  $-\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$  is negative or positive, i.e., as  $ax_1 + by_1 + c$  and  $ax_2 + by_2 + c$  are of the same or opposite signs.

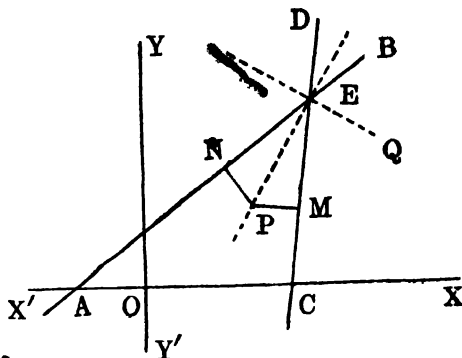
**Cor.** The point  $P_1(x_1, y_1)$  is on the same side of the line  $ax + by + c = 0$  as the origin if  $ax_1 + by_1 + c$  is of the same sign as of  $c$ .

**3.11. Equations of bisectors of the angles between two given straight lines**

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0.$$

Any point  $x_1, y_1$  (say) on any of the bisectors must be equi-

distant from the two lines, so that the perpendicular distances from it to the two lines must be equal in magnitude. If it be on the bisector of the angle in which the origin lies (as at  $P$ ) both perpendiculars must be of the same sign, because the point and the origin lie on the same side with respect to each of the lines. On the other hand, if it be on the bisector of the angle in which the origin does not lie (as at  $Q$ ) clearly it must be on the same side of the origin with respect to one of the lines ( $AB$ ), and on opposite side of the origin with respect to the other line ( $CD$ ). Hence, the two perpendiculars, being equal in magnitude, must be opposite in sign.



Hence, for points on one of the bisectors,

$$\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

and for points on the other,

$$\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Hence, the equations to the bisectors of the angles between the two given lines (i.e., the locus of  $x_1, y_1$ ) are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

**Note.** If  $c_1$  and  $c_2$  be of the same sign, the + sign on the right-hand side will give the bisector of the angle in which the origin lies, and - sign will give the other bisector.

If  $c_1$  and  $c_2$  be of opposite signs, the - sign on the right-hand side will give the bisector of the angle containing the origin, and + sign will give the other bisector.

**3'12. Illustrative Examples.**

**Ex. 1.** Find the perpendicular distance from the point  $(-2, -7)$  to the straight line  $7y - 24x = 10$ . Is the point on the same side of the line as the origin, or on the opposite side?

The equation of the line can be written as

$$24x - 7y + 10 = 0.$$

The perpendicular distance from  $(-2, -7)$  on it is given by

$$p = \frac{24(-2) - 7(-7) + 10}{\pm \sqrt{24^2 + 7^2}}.$$

The proper sign to be chosen for the denominator here is + (same as the sign of the constant term +10).

$$\text{Hence, } p = \frac{-48 + 49 + 10}{+25} = \frac{11}{25}.$$

As  $24(-2) - 7(-7) + 10$  (i.e., +11) has the same sign as of the constant term +10, the point  $(-2, -7)$  is on the same side of the given line as the origin.

The perpendicular from the point on the line is thus to be associated with positive sign, as has been found to be the case.

**Ex. 2.** Find the bisector of the angle containing the origin between the lines  $4x - 3y + 5 = 0$  and  $5x - 12y - 2 = 0$ .

The bisectors of the angles between the given lines are [ See § 3'11 ],

$$\frac{4x - 3y + 5}{\sqrt{4^2 + 3^2}} = \pm \frac{5x - 12y - 2}{\sqrt{5^2 + 12^2}}.$$

Of these, (by note of § 3'11), the - sign on the right-hand will give the bisector of the angle containing the origin (∵ here 5 and -2 are of opposite signs).

Thus, the required bisector is

$$\frac{4x - 3y + 5}{5} = - \frac{5x - 12y - 2}{13},$$

$$\text{or, } 77x - 99y + 55 = 0, \text{ i.e., } 7x - 9y + 5 = 0.$$

**Ex. 3.** If  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  be the co-ordinates of the vertices

of a triangle, and  $a, b, c$  respectively be the lengths of the sides opposite to these vertices, then the co-ordinates of the in-centre of the triangle are

$$\frac{ax_1 + bx_2 + cx_3}{a+b+c} \text{ and } \frac{ay_1 + by_2 + cy_3}{a+b+c}.$$

Let  $A, B, C$  denote the vertices having co ordinates  $(x_1, y_1), (x_2, y_2)$  and  $(x_3, y_3)$  respectively;  $BC=a, CA=b, AB=c$ .

Let  $AD$  be the bisector of the angle  $BAC$ , meeting  $BC$  at  $D$ ; then we know from geometry that

$$BD \cdot DC = AB \cdot AC = c \cdot b.$$

$\therefore$  the co-ordinates of  $D$  are [ See § I'3 ]

$$\frac{cx_2 + bx_3}{c+b}, \frac{cy_2 + by_3}{c+b}.$$

$$\text{Also, } \frac{BD}{DC} = \frac{c}{b} \therefore \frac{BD}{BC} = \frac{c}{b+c}. \text{ Hence, } BD = \frac{c}{b+c}$$

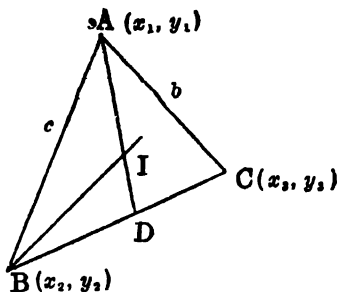
Now,  $BI$  being the bisector of the angle  $ABC$ , meeting  $AD$  at  $I, I$  is the in centre of the triangle  $ABC$ .

$$\text{Also, } \frac{DI}{IA} = \frac{BD}{DA} = \frac{ac}{b+c} : c = a : (b+c).$$

Thus, the co ordinates of  $D$  and  $A$  being known, the co-ordinates of  $I$  are

$$\frac{a \cdot x_1 + (b+c) \cdot \frac{cx_2 + bx_3}{c+b}}{a + (b+c)} \quad \text{and} \quad \frac{a \cdot y_1 + (b+c) \cdot \frac{cy_2 + by_3}{c+b}}{a + (b+c)}$$

$$\text{i.e., } \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}.$$



### Examples III(b)

1. Find the perpendicular distance (i) from the point  $(-1, -3)$  to the straight line  $5x - 12y + 8 = 0$ , (ii) from the point  $(a, b)$  to the straight line  $\frac{x}{a} - \frac{y}{b} = 1$ .

2. Show that the distance of the point  $(-5, 2)$  from the point  $(-1, -1)$  is equal to its perpendicular distance from the straight line  $4(y-1) = 3(x-2)$ .

3. Show that the point  $(-3, 1)$  is equidistant from the two lines  $4x - 3y = 2$  and  $24x + 7y = 20$ .

4. Find the perpendicular distance of the point of intersection of the two lines  $2x + 3y + 6 = 0$  and  $5x - y - 19 = 0$  from the line  $6x - 8y = 0$ . What are the co-ordinates of the foot of the perpendicular?

5. Find the points on the  $y$ -axis whose perpendicular distance from the straight line  $5y = 12x - 16$  is 2.

6. Find the distance between the parallel lines  $5x - 4y + 1 = 0$  and  $3(x-2) = 4y - 3$ .

7. (i) Show that the product of the perpendiculars from the two points  $(\pm 4, 0)$  on the straight line  $3x \cos \theta + 5y \sin \theta = 15$  is independent of  $\theta$ .

(ii) Show that the product of the perpendiculars drawn from the two points  $(\pm \sqrt{a^2 - b^2}, 0)$  upon the straight line

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \text{ is } b^2.$$

8. (i) Find the perpendicular distance of the point  $(2, 2)$  from the line joining the points  $(-5, 1)$  and  $(3, -5)$ .

(ii) Prove that the points  $(2, -6)$  and  $(-4, -12)$  are equidistant from the line joining  $(1, -4)$  and  $(-3, -14)$ , and are on opposite sides of it.

9. Show that the points  $(2, -3)$  and  $(-4, -2)$  are on opposite sides of the line  $5x - 7y - 11 = 0$ . Which of the points is on the same side of the line as the origin?

10. Prove that the origin is inside the triangle whose vertices are  $(2, 1)$ ,  $(3, -2)$ ,  $(-4, -1)$ .

[The origin lies on the same side with each vertex with respect to the line joining the other two.]

11. Find the bisector of the angle between the lines  $12x + 5y = 4$  and  $3x + 4y + 7 = 0$  containing the origin.

12. Find the bisectors of the angles between the lines  $24x - 7y - 2 = 0$  and  $3y - 4x + 7 = 0$ , and verify that these bisectors are at right angles. Which of these bisects the angle in which the origin lies?

13. Find the in-centre of the triangle

(i) whose vertices are  $(-1, -2)$ ,  $(-1, 3)$  and  $(11, -2)$ .

[As in Ex. 3, § 3.12, let  $A, B, C$  denote the vertices having co-ordinates  $(-1, -2)$ ,  $(-1, 3)$ ,  $(11, -2)$  respectively.

$$\text{Then } AB = \sqrt{(-1+1)^2 + (3+2)^2} = 5; \quad BC = \sqrt{(11+1)^2 + (-2-3)^2} = 13$$

$$\text{and } CA = \sqrt{(11+1)^2 + (-2+2)^2} = 12.$$

Let  $AD$  be the bisector of the  $\angle BAC$  meeting  $BC$  at  $D$ . Then  $\frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{12}$ .  $\therefore$  Co-ordinates of  $D$   $\left( \frac{5 \cdot 11 + 12(-1)}{5+12}, \frac{5(-2) + 12 \cdot 3}{5+12} \right)$

$$\text{i.e., } \left( \frac{47}{17}, \frac{26}{17} \right).$$

$$\text{Also } \frac{BD}{DC} = \frac{5}{12}. \quad \therefore \frac{BD}{BC} = \frac{5}{17}. \quad \therefore BD = \frac{5}{17} \cdot BC = \frac{5}{17} \cdot 13 = \frac{65}{17}.$$

Now  $BI$  being the bisector of  $\angle ABC$  meeting  $AD$  at  $I$ ,  $I$  is the incentre of the  $\triangle ABC$ .

$$\therefore \frac{DI}{IA} = \frac{BD}{BA} = \frac{\frac{65}{17}}{5} = \frac{13}{17}.$$

$$\therefore \text{Co-ordinates of } I \text{ are } \left( \frac{13(-1) + 17 \cdot \frac{47}{17}}{13+17}, \frac{13(-2) + 17 \cdot \frac{26}{17}}{13+17} \right) \text{ i.e., } (1, 0).$$

(ii) whose sides are  $x = 3$ ,  $y = -4$  and  $3x + 4y = 17$ .

[Solving the equations pair-wise let the co-ordinates of  $A, B, C$  respectively are  $(3, -4)$ ;  $(11, -4)$ ;  $(3, 2)$ .

$$\text{As in Ex. 13 (i), } AB = \sqrt{(11-3)^2 + (-4+4)^2} = 8;$$

$$BC = \sqrt{(11-3)^2 + (-4-2)^2} = 10; \quad CA = \sqrt{(3-3)^2 + (-4-2)^2} = 6.$$

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{8}{6} = \frac{4}{3}. \quad \therefore \text{Co-ordinates of } D \text{ are } \left( \frac{4 \cdot 3 + 3 \cdot 11}{4+3}, \frac{4 \cdot (-4) + 3 \cdot 2}{4+3} \right)$$

$$\text{i.e., } \left( \frac{45}{7}, -\frac{4}{7} \right).$$

$$\therefore \frac{BD}{DC} = \frac{4}{3}. \quad \therefore \frac{BD}{BC} = \frac{4}{7}. \quad \therefore BD = \frac{4}{7} \cdot BC = \frac{4}{7} \cdot 10 = \frac{40}{7}.$$

Now,  $\frac{DI}{IA} = \frac{BD}{BA} = \frac{4^2}{8} = \frac{5}{7}$ .

$\therefore$  Co-ordinates of ~~the~~ are  $\left(\frac{5 \cdot 3 + 7 \cdot 4^2}{5+7}, \frac{5(-4) + 7(-\frac{4}{3})}{5+7}\right)$ , i.e.,  $(5, 2)$ .]

14. The algebraic sum of the perpendicular distances on a straight line from the three given points  $(5, -3)$ ,  $(-2, 4)$  and  $(-3, -7)$  is zero. Show that the straight line passes through a fixed point.

[ Let the equation of the line be  $ax+by+c=0$ .

Then  $\frac{5a-3b+c}{\pm\sqrt{a^2+b^2}} + \frac{-2a+4b+c}{\pm\sqrt{a^2+b^2}} + \frac{-3a-7b+c}{\pm\sqrt{a^2+b^2}} = 0$ ,

or,  $5a-3b+c-2a+4b+c-3a-7b+c=0$ , or,  $-6b+3c=0$ .

$$\frac{b}{c} = \frac{1}{2}.$$

Now  $ax+by+c=0$  can always be written as  $\frac{a}{c}x + \frac{b}{c}y + 1 = 0$ . But whatever be the value of  $a$ , the sum of perpendicular lengths remains unchanged.

Hence  $a=0$  or  $\frac{b}{c} = \frac{1}{2}$ .

Therefore the required equation becomes  $\frac{1}{2}y+1=0$  or  $y=-2$ .

Hence the straight line always passes through  $(0, -2)$ .

N.B. For example straight line passing through  $(0, -2)$  is  $y+2=m(x-0)$  or  $mx-y-2=0$ . And when we consider the sum of the  $\perp$  length, it becomes  $\frac{1}{\pm\sqrt{m^2+1}} (5m+3-2-2m-4-2-3m+7-2)=0$ .]

### ANSWERS

1. (i) 3. (ii)  $\frac{ab}{\sqrt{a^2+b^2}}$ . 4. 5;  $(0, 0)$ . 5.  $(0, 2)$  and  $\left(0, -\frac{42}{5}\right)$ .

6. 2. 8. (i) 5. 9.  $(-4, -2)$ . 11.  $99x+77y+71=0$ .

12.  $44x-22y-37=0$ ,  $4x+8y+33=0$ ; latter. 13. (i)  $(1, 0)$ . (ii)  $(5, -2)$ .

# **Higher Secondary Syllabus of Elective Mathematics :**

## **CO-ORDINATE GEOMETRY**

### **( Course for Class XI )**

Circles, chords, tangents ; Normals and elementary properties connected with them ; Parabola, Ellipse, Hyperbola referred to their principal axes ; Analytical treatment of these curves in respect of (1) the focus and directrix properties, (2) tangents and normals and elementary properties connected with them, (3) centre and diameter.

[ *Note*—Discussion should always be restricted to rectangular cartesian co-ordinates. ]



## CONTENTS

| CHAPTER      |     |     | PAGE |
|--------------|-----|-----|------|
| 4. Circle    | ... | ... | 53   |
| 5. Conics    | ... | ... | 73   |
| 6. Parabola  | ... | ... | 80   |
| 7. Ellipse   | ... | ... | 100  |
| 8. Hyperbola | ... | ... | 121  |

## IMPORTANT FORMULÆ & RESULTS

### H. S. Co-ordinate Geometry ( Class XI )

#### 1. Equation of the circle

(i) standard form :  $x^2 + y^2 = a^2$ .

centre :  $(0, 0)$  ; radius  $a$ .

(ii) general form :  $x^2 + y^2 + 2gx + 2fy + c = 0$

centre :  $(-g, -f)$ , radius =  $\sqrt{g^2 + f^2 - c}$ .

2. Circle with the given points  $(x_1, y_1)$  and  $(x_2, y_2)$  as extremities of a diameter  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ .

#### 3. Equation of the tangent to the circle at $(x_1, y_1)$

(i) for standard form :  $xx_1 + yy_1 = a^2$ ,

(ii) for general form :

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

#### 4. Equation of the normal to the circle at $(x_1, y_1)$

(i) for standard form :  $\frac{x}{x_1} = \frac{y}{y_1}$ .

(ii) for general form :  $x(y_1 + f) - y(x_1 + g) = fx_1 - gy_1$ .

5. Length of the chord of the circle  $x^2 + y^2 = a^2$  intercepted by the line  $y = mx + c$  is

$$2 \frac{\sqrt{a^2(1+m^2) - c^2}}{\sqrt{1+m^2}}.$$

6. Condition of tangency : condition that the line  $y = mx + c$  may touch the circle  $x^2 + y^2 = a^2$  is  $c = \pm a\sqrt{1+m^2}$

$y = mx + a\sqrt{1+m^2}$  is a tangent to the circle  $x^2 + y^2 = a^2$  for all values of  $m$ , and in that case the point of contact is

$$-\frac{am}{\sqrt{1+m^2}}, \quad \frac{a}{\sqrt{1+m^2}}$$

7. Length of the tangent from an external point  $(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

## 8. Standard forms of the equations of conics.

## (a) Parabola

(i)  $y^2 = 4ax$  (with axis and directrix as axes of co-ordinates).(ii)  $y^2 = 4ax$  (standard form),

(with the vertex as origin and the axis and the tangent at the vertex as axes of co-ordinates).

## (b) Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (\text{standard form}).$$

(with centre as origin, and major and minor axes as axes of co-ordinates)

## (c) Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{standard form})$$

(with centre as origin and transverse and conjugate axes as axes of co-ordinates).

## 9. Parabola :

(i) Standard form  $y^2 = 4ax$ .(ii) Latus rectum  $= 4a$  ; focus is  $(a, 0)$  ; extremities of the latus rectum are  $(a, \pm 2a)$  ; directrix is  $x = -a$ .(iii) Equation of the tangent at  $(x_1, y_1)$  is  $yy_1 = 2a(x + x_1)$ .(iv) Normal at  $(x_1, y_1)$  is  $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ .

(v) Length of the chord intercepted by the straight line

$$y = mx + c \text{ is } \frac{4}{m^2} \sqrt{a(a - mc)(1 + m^2)}.$$

(vi) Condition that  $y = mx + c$  may touch the

$$\text{parabola is } c = \frac{a}{m} \quad (m \neq 0).$$

The line  $y = mx + \frac{a}{m}$  is a tangent to the parabola for all values of  $m$  (except zero),

the point of contact being  $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ .

(vii) Parametric representation :  $x = at^2, y = 2at$ .

(viii) Equation of the diameter :  $y = \frac{2a}{m}x$ .

## 10. Ellipse

(i) Standard form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

(ii) Latus rectum  $= 2a(1 - e^2) = 2 \frac{b^2}{a}$ .

(iii) Eccentricity :  $b^2 = a^2(1 - e^2)$  or  $e^2 = \frac{a^2 - b^2}{a^2}$ .

(iv) Focal distances of  $P(x_1, y_1)$  :

$$SP = a - ex_1, S'P = a + ex_1; SP + S'P = 2a.$$

(v) Tangent at  $(x_1, y_1)$  :  $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ .

(vi) Normal at  $(x_1, y_1)$  :  $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{\frac{y_1}{b^2}}$ .

(vii) Length of the chord intercepted by the line

$$y = mx + c \text{ on the ellipse} \\ = \frac{2ab \sqrt{1 + m^2} \sqrt{a^2 m^2 + b^2 - c^2}}{a^2 m^2 + b^2}.$$

(viii) Condition of tangency :

The line  $y = mx + c$  is a tangent to the ellipse if

$$c = \pm \sqrt{a^2 m^2 + b^2}.$$

The line  $y = mx + \sqrt{a^2 m^2 + b^2}$  is a tangent to the ellipse for all values of  $m$ , and the point of contact is

$$-\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \frac{b^2}{\sqrt{a^2 m^2 + b^2}}.$$

(ix) Auxiliary circle :  $x^2 + y^2 = a^2$ .

(x) Parametric representation :  $x = a \cos \theta, y = b \sin \theta$ .

(xi) Diameter  $y = -\frac{b^2}{a^2 m}x$ .

(xii) Director circle  $x^2 + y^2 = a^2 + b^2$ .

## 11. Hyperbola

(i) Standard Equation :  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

(ii) Latus rectum :  $= 2a(e^2 - 1) = 2\frac{b^2}{a}.$

(iii) Eccentricity :  $b^2 = a^2(e^2 - 1)$  or  $e^2 = \frac{a^2 + b^2}{a^2}.$

For rectangular (or equilateral) hyperbola

$$a = b; e = \sqrt{2}.$$

(iv) Focal distances of  $P(x_1, y_1)$

$$SP = ex_1 - a, S'P = ex_1 + a$$

$$S'P - SP = 2a.$$

(v) Equation of the tangent at  $(x_1, y_1)$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(vi) Equation of the normal at  $(x_1, y_1)$  is

$$\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}.$$

(vii) Length of the chord of the hyperbola intercepted by  $y = mx + c$  is

$$\frac{2ab\sqrt{1+m^2}\sqrt{c^2 - a^2m^2 + b^2}}{a^2m^2 - b^2}.$$

(viii) Condition of tangency :

The line  $y = mx + c$  will be a tangent to the hyperbola if  $c = \pm \sqrt{a^2m^2 - b^2}.$

The line  $y = mx + \sqrt{a^2m^2 - b^2}$  is a tangent to the hyperbola for all values of  $m$ , the point of contact

being  $\left( -\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, -\frac{b^2}{\sqrt{a^2m^2 - b^2}} \right).$

(ix) Equation of the diameter is  $y = \frac{b^2}{a^2m}x.$

(x) Equation of the asymptotes :  $y = \pm \frac{b}{a}x.$

# CO-ORDINATE GEOMETRY

( To be taught in Class XI )

## CHAPTER IV

### CIRCLE

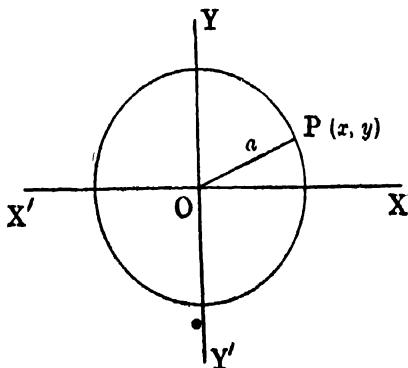
#### 4.1. Circle of given radius, with centre at the origin.

Let  $a$  be the radius of a circle whose centre is at the origin  $O$ . If  $P(x, y)$  be any point on the circle, the distance  $OP = a$ ,

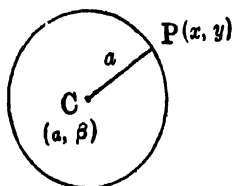
$$\text{or, } OP^2 = a^2 ;$$

$$x^2 + y^2 = a^2$$

This being the relation satisfied by the co-ordinates  $(x, y)$  of any point on the circle, it represents the equation to the circle.



#### 4.2. Circle of given radius, with centre at any given point.



Let  $C(a, \beta)$  be the centre, and  $a$  the radius of a circle. If  $P(x, y)$  be any point on the circle, the distance  $CP = a$ , or  $CP^2 = a^2$ ;

$$\therefore (x - a)^2 + (y - \beta)^2 = a^2.$$

This being the relation satisfied by the co-ordinates  $(x, y)$  of any point on the circle, it represents the equation to the circle.

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**Note.** From above it is clear that any circle, with centre anywhere (say  $\alpha, \beta$ ) and radius of any length (a say) has equation of the form

$$x^2 + y^2 - 2\alpha x - 2\beta y + (\alpha^2 + \beta^2 - a^2) = 0,$$

i.e., of the form  $x^2 + y^2 + 2gx + 2fy + c = 0$

where  $g, f, c$  are constants.

This is thus the most general form of the equation to a circle.

[ See also § 4.3 and its note ]

For a circle with centre as origin,  $g$  and  $f$  are zeroes.

**4.3.** To show that the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

always represents a circle (for all values of the constants  $g, f, c$ ), and to find its centre and radius.

The given equation can be written in the form

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c,$$

$$\text{or, } (x + g)^2 + (y + f)^2 = g^2 + f^2 - c,$$

$$\text{or, } \{x - (-g)\}^2 + \{y - (-f)\}^2 = (\sqrt{g^2 + f^2 - c})^2.$$

This shows that the distance of the variable point  $(x, y)$  from the fixed point  $(-g, -f)$  is a constant  $= \sqrt{g^2 + f^2 - c}$ .

Hence, the locus represented by the equation is a circle, with centre  $(-g, -f)$ , and radius  $\sqrt{g^2 + f^2 - c}$ .

**Note.** The equation may be multiplied by a constant factor  $a$ , and put in the form

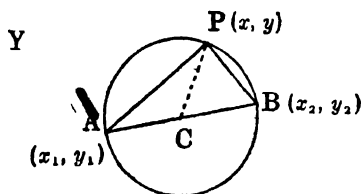
$$ax^2 + ay^2 + 2g'x + 2f'y + c' = 0 \quad \dots \quad \dots \quad (i)$$

This also represents a circle, but the centre here is not  $(-g', -f')$ , nor is the radius  $= \sqrt{g'^2 + f'^2 - c'}$ . In fact, whenever in a second degree equation (in rectangular co-ordinates) the coefficients of  $x^2$  and  $y^2$  are equal, and there is no term involving the product  $xy$ , the equation represents a circle. The equation (i) is thus the most general form of the equation of the circle. To get its centre and radius, we are to divide out the equation by the common coefficient  $a$  of  $x^2$  or  $y^2$ , and reduce it to the form  $x^2 + y^2 + 2gx + 2fy + c = 0$ .

Then  $(-g, -f)$  are the co-ordinates of the centre and  $\sqrt{g^2 + f^2 - c}$  is the length of the radius.

**4.4. Circle with the given points  $(x_1, y_1)$  and  $(x_2, y_2)$  as extremities of a diameter.**

$A(x_1, y_1)$  and  $B(x_2, y_2)$  being the extremities of a diameter, the mid-point of  $AB$ , whose co-ordinates are  $\frac{1}{2}(x_1 + x_2)$ ,  $\frac{1}{2}(y_1 + y_2)$  is the centre. Also the radius  $= \frac{1}{2}AB = \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ .



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Hence, the equation to the circle is

$$\begin{aligned} \{x - \tfrac{1}{2}(x_1 + x_2)\}^2 + \{y - \tfrac{1}{2}(y_1 + y_2)\}^2 \\ = \tfrac{1}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\} \end{aligned} \quad \dots \quad (i)$$

*Alternatively*

$P$  being any point  $(x, y)$  on the circle,  $AB$  being a diameter,  $PA$  and  $PB$  must be at right angles. Now 'm' of  $PA$  is  $\frac{y - y_1}{x - x_1}$  and that of  $PB$  is  $\frac{y - y_2}{x - x_2}$ . [ See § 3'1(E) ]

Hence, for  $PA$  and  $PB$  to be at right angles,

$$\frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1,$$

$$\text{or,} \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \quad (ii)$$



which being the equation satisfied by the co-ordinates of any point  $P$  on the circle, it represents the required equation to the circle.

**Note.** It may be noted that the forms (i) and (ii) are identical, as can be shown by simplifying.

**4.5. Circle passing through any three given points**  
 $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ .

$$\text{Let } x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots \quad (i)$$

be the equation to the circle.

As it passes through the three given points,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

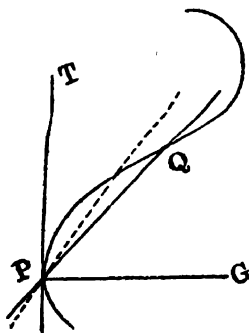
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0.$$

From these three equations, which are linear equations in the unknowns  $g, f, c$ , we get definite values of these unknowns when  $(x_1, y_1), (x_2, y_2)$  and  $(x_3, y_3)$  are given.

Substituting these values of  $g, f, c$  in (i), we get the required equation of the circle, as also its centre  $(-g, -f)$  and radius  $\sqrt{g^2 + f^2 - c}$ .

#### 4.6. Definitions of Tangent and Normal.



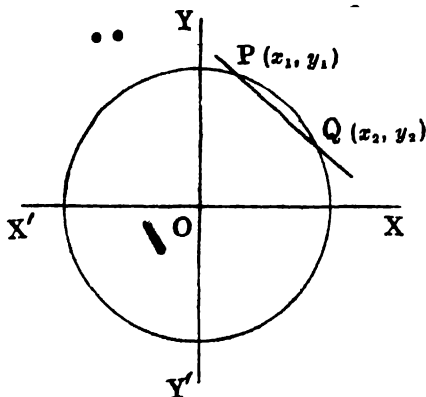
If  $P$  be any point on a curve, and if we take a neighbouring point  $Q$  on it, then if the line joining  $P$  and  $Q$  be turned about  $P$  so that the other point of intersection  $Q$  gradually approaches  $P$ , the limiting position  $PT$  of the line  $PQ$ , when  $Q$  ultimately coincides with  $P$ , is defined as the *tangent* to the curve at  $P$ .

The line  $PG$  through  $P$ , perpendicular to the tangent at  $P$ , is defined as the *normal* to the curve at  $P$ .

4.7. Equation to the tangent at a given point  $(x_1, y_1)$  on the circle :

(A)  $x^2 + y^2 = a^2$ .

(B)  $x^2 + y^2 + 2gx + 2fy + c = 0$ .



(A) Let  $P$  be the point  $(x_1, y_1)$  on the circle

$$x^2 + y^2 = a^2, \quad \text{... (i)}$$

and let  $Q(x_2, y_2)$  be a neighbouring point on it.

The equation to the chord  $PQ$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad \text{... (ii)}$$

Now, since  $P$  and  $Q$  both lie on the circle (i), we have

$$x_1^2 + y_1^2 = a^2 \quad \text{... (ii)}$$

$$x_2^2 + y_2^2 = a^2 \quad \text{... (iv)}$$

$\therefore$  subtracting,

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) = 0,$$

whence  $\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}$ .

∴ equation (ii) can be written as

$$y - y_1 = -\frac{x_2 + x_1}{y_2 + y_1} (x - x_1). \quad \dots \quad \dots \quad (v)$$

Now make  $Q$  approach  $P$  and ultimately coincide with it, so that the co-ordinates  $(x_2, y_2)$  coincide with  $(x_1, y_1)$ . In that limiting position, the chord  $PQ$  becomes the tangent at  $P$ , whose equation, [from (v)] then becomes

$$y - y_1 = -\frac{2x_1}{2y_1} (x - x_1),$$

$$\text{or, } x_1(x - x_1) + y_1(y - y_1) = 0,$$

$$\text{i.e., } xx_1 + yy_1 = x_1^2 + y_1^2 = a^2 \quad [\text{by (iii)}].$$

Hence, the equation of the tangent at  $(x_1, y_1)$  is

$$xx_1 + yy_1 = a^2.$$

(B) Let  $P$  be the point  $(x_1, y_1)$  on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad \dots \quad \dots \quad (i)$$

and let  $Q(x_2, y_2)$  be a neighbouring point on it. The equation of the chord  $PQ$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad \dots \quad \dots \quad (ii)$$

Now, since  $P$  and  $Q$  both lie on the circle (i), we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots \quad (iii)$$

$$\text{and } x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \dots \quad (iv)$$

∴ subtracting,

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0,$$

$$\text{whence, } (x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0,$$

$$\text{or, } \frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}.$$

Hence, equation (ii) of chord  $PQ$  can be written as

$$y - y_1 = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f} (x - x_1).$$

Now making  $Q$  approach  $P$  and ultimately coincide with it [ so that  $(x_2, y_2)$  coincide with  $(x_1, y_1)$  ], the chord becomes the tangent at  $P$ , whose equation is then

$$y - y_1 = -\frac{2(x_1 + g)}{2(y_1 + f)}(x - x_1),$$

$$\text{or, } (x_1 + g)(x - x_1) + (y_1 + f)(y - y_1) = 0,$$

$$\text{or, } xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1,$$

or, adding  $gx_1 + fy_1 + c$  to both sides,

$$\begin{aligned} xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c \\ = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \quad [\text{by (iii)}] \end{aligned}$$

Hence, the tangent to the circle (i) at  $(x_1, y_1)$  is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

4.8. Equation to the normal at  $(x_1, y_1)$  to the circle :

$$(A) \ x^2 + y^2 = a^2.$$

$$(B) \ x^2 + y^2 + 2gx + 2fy + c = 0.$$

(A) The tangent at  $(x_1, y_1)$  to the circle  $x^2 + y^2 = a^2$  is  $xx_1 + yy_1 = a^2$ , or,  $y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1}$ , of which the 'm' is  $-\frac{x_1}{y_1}$ .

The normal, which is perpendicular to the tangent, through  $(x_1, y_1)$  is then

$$y - y_1 = \frac{y_1}{x_1}(x - x_1),$$

$$\text{or, } \frac{x - x_1}{x_1} = \frac{y - y_1}{y_1}, \text{ or, } \frac{x}{x_1} = \frac{y}{y_1},$$

which evidently passes through the origin, i.e., the centre of the circle.

(B) The tangent at  $(x_1, y_1)$  to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$\text{is } xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

## CO-ORDINATE GEOMETRY

$$\text{i.e., } x(x_1 + g) + y(y_1 + f) + (gx_1 + fy_1 + c) = 0$$

of which the 'm' is  $-\frac{x_1 + g}{y_1 + f}$

The normal, which is perpendicular to the tangent, through  $(x_1, y_1)$  is then

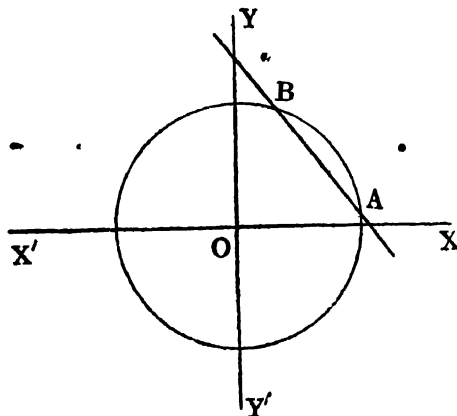
$$y - y_1 = \frac{y_1 + f}{x_1 + g} (x - x_1),$$

$$\text{or, } x(y_1 + f) - y(x_1 + g) = fx_1 - gy_1.$$

**Note.** Evidently this line passes through the point  $(-g, -f)$ , which is the centre of the circle.

Thus, the normal at any point of a circle passes through its centre, in other words, the radius to any point of a circle is perpendicular to the tangent at the point.

**4'9. Length of the chord of the circle  $x^2 + y^2 = a^2$ , intercepted by the straight line  $y = mx + c$ .**



At the points of intersection of the line with the circle both the equations are satisfied. Hence, eliminating  $y$  between the two equations, the abscissæ of the points of intersection will be given by

$$x^2 + (mx + c)^2 = a^2,$$

$$\text{or, } x^2(1 + m^2) + 2mcx + (c^2 - a^2) = 0, \quad \dots (i)$$

which being a quadratic equation in  $x$ , there are only two values of  $x$  and accordingly only two points of intersection of the straight line with the circle (which may be real and distinct, real and coincident, or imaginary).

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the two points  $A$  and  $B$  of intersection. Then  $x_1$  and  $x_2$  are the roots of (i).

$$\therefore x_1 + x_2 = -\frac{2mc}{1+m^2} \text{ and } x_1x_2 = \frac{c^2 - a^2}{1+m^2}.$$

$$\begin{aligned} \therefore (x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= \frac{4m^2c^2}{(1+m^2)^2} - \frac{4(c^2 - a^2)}{1+m^2} \\ &= \frac{4\{m^2c^2 - (c^2 - a^2)(1+m^2)\}}{(1+m^2)^2} \\ &= \frac{4\{a^2(1+m^2) - c^2\}}{(1+m^2)^2}. \end{aligned}$$

Again,  $y_1 = mx_1 + c$  and  $y_2 = mx_2 + c$ .

$$\therefore y_1 - y_2 = m(x_1 - x_2).$$

Length of the chord  $AB$

$$\begin{aligned} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2 (1+m^2)} \\ &= \sqrt{\frac{4\{a^2(1+m^2) - c^2\}}{1+m^2}} = 2 \frac{\sqrt{a^2(1+m^2) - c^2}}{\sqrt{1+m^2}}. \end{aligned}$$

**Cor. Condition of tangency.**

The given line will touch the circle if the chord intercepted is zero. Hence the condition that the given line  $y = mx + c$  may touch the circle  $x^2 + y^2 = a^2$  is  $c^2 = a^2(1+m^2)$ ,

$$\text{or, } c = \pm a\sqrt{1+m^2}.$$

*Alternative method for condition of tangency.*

The perpendicular from the centre of the circle to the line is equal to the radius.

$\therefore$  perpendicular from  $O$ ,  $O$  on  $mx - y + c = 0$  is equal to  $a$ ,

$$\text{or, } \frac{c}{\sqrt{1+m^2}} = a;$$

$$\therefore c = \pm a\sqrt{1+m^2}.$$

**Note.** There are thus two tangents to the circle parallel to a given line  $y = mx + c$ , (i.e., with a given  $m$ ), namely,

$$y = mx \pm a\sqrt{1+m^2}.$$

**4.10.** To show that  $y = mx + a\sqrt{1+m^2}$  is a tangent to the circle  $x^2 + y^2 = a^2$ , and to find the point of contact.

The tangent at  $(x_1, y_1)$  of the circle is

$$xx_1 + yy_1 = a^2 \quad \text{or} \quad xx_1 + yy_1 - a^2 = 0 \quad \dots \quad (i)$$

$$\text{If the line } y = mx + a\sqrt{1+m^2} \quad \text{or} \quad mx - y + a\sqrt{1+m^2} = 0 \quad \dots \quad (ii)$$

be a tangent to the circle at  $(x_1, y_1)$ , the equation (i) and (ii) must be the same. Hence, comparing coefficients,

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{-a^2}{a\sqrt{1+m^2}} = \frac{-a}{\sqrt{1+m^2}}.$$

$$\therefore \quad x_1 = -\frac{am}{\sqrt{1+m^2}}, \quad y_1 = \frac{a}{\sqrt{1+m^2}}.$$

The line (ii) therefore will touch the circle only if the assumed point  $(x_1, y_1)$  is really a point on the circle,

$$\text{i.e., if } \left(-\frac{am}{\sqrt{1+m^2}}\right)^2 + \left(\frac{a}{\sqrt{1+m^2}}\right)^2 = a^2$$

which is evidently satisfied.

Thus,  $y = mx + a\sqrt{1+m^2}$  touches the circle, whatever  $m$  may be and the point of contact is given by

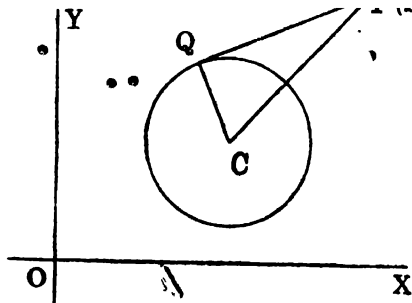
$$x_1 = -\frac{am}{\sqrt{1+m^2}}, \quad y_1 = \frac{a}{\sqrt{1+m^2}}.$$

**Note.** Similarly,  $y = mx - a\sqrt{1+m^2}$  is also a tangent to the circle  $x^2 + y^2 = a^2$ , the point of contact being  $\frac{am}{\sqrt{1+m^2}}, -\frac{a}{\sqrt{1+m^2}}$ .

**4.11.** Length of the tangent from an external point  $(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$ .

Let  $P$  be the point  $(x_1, y_1)$  and  $PQ$  a tangent to the circle from  $P$ . The centre  $C$  of the circle is  $(-g, -f)$ , and the radius  $CQ$  is  $\sqrt{g^2 + f^2 - c}$ .

Now since, as proved before (§ 4'8, *Note*),  $CQ$  is perpendicular to  $CP$ ,



$$\begin{aligned} PQ^2 &= CP^2 - CQ^2 \\ &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

$$\therefore PQ = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

**Cor.** The length of the tangent from the external point  $(x_1, y_1)$  to the circle  $x^2 + y^2 = a^2$  is  $\sqrt{x_1^2 + y_1^2 - a^2}$ .

**Note.** We notice that when the co-ordinates  $(x_1, y_1)$  of any point are substituted for  $(x, y)$  on the left-hand side of the equation to a circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  or  $x^2 + y^2 - a^2 = 0$ , we obtain the square on the length of the tangent from the point  $(x_1, y_1)$  to the circle. If this is positive, the point is outside the circle, and the length of the tangent is real. If the expression is negative, the point is inside the circle, and the length of the tangent is imaginary.

If the equation to the circle be in the form  $ax^2 + ay^2 + 2g'x + 2f'y + c' = 0$ , we are first to divide out the equation by  $a$ , and reduce it to the form  $x^2 + y^2 + 2gx + 2fy + c = 0$  and then substitute  $(x_1, y_1)$  on the left side to get the square on the length of the tangent. [See § 4'3 note in this connection.]

#### 4'12. Illustrative Examples.

**Ex. 1.** Find the equation to the circle passing through the points  $(2, -3)$  and  $(-3, -4)$  and having its centre on the line  $7x + 2y + 6 = 0$ .



Let  $(\alpha, \beta)$  be the co-ordinates of the centre of the circle. As the circle passes through the points  $(2, -3)$  and  $(-3, -4)$ , these points must be equidistant from the centre. Hence,

$$(\alpha - 2)^2 + (\beta + 3)^2 = (\alpha + 3)^2 + (\beta + 4)^2,$$

$$\text{or, } 10\alpha + 2\beta + 12 = 0, \text{ i.e., } 5\alpha + \beta + 6 = 0.$$

Also, since the centre lies on the given line, we have

$$7\alpha + 2\beta + 6 = 0. \quad \dots \quad \dots \quad \dots \quad \text{(ii)}$$

From (i) and (ii), solving,  $\alpha = -2$ ,  $\beta = 4$ . The radius of the circle,  $r$ , is equal to the distance of the point  $(2, -3)$  from the centre  $(-2, 4)$

$$\therefore r^2 = (2 + 2)^2 + (-3 - 4)^2 = 65.$$

Hence, the required equation to the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2, \text{ or, } (x + 2)^2 + (y - 4)^2 = 65,$$

$$\text{or, } x^2 + y^2 + 4x - 8y - 45 = 0.$$

**Ex. 2.** Find the length of the chord intercepted by the straight line  $3x - 4y + 5 = 0$  of the circle passing through the points  $(1, 2)$ ,  $(3, -4)$  and  $(5, -6)$ .

$$\text{Let } x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

be the equation to the circle passing through the given points  $(1, 2)$ ,  $(3, -4)$ , and  $(5, -6)$ .

Then, substituting values of the co-ordinates,

$$5 + 2g + 4f + c = 0$$

$$25 + 6g - 8f + c = 0$$

$$\text{and } 61 + 10g - 12f + c = 0$$

Solving these, we get  $g = -11$ ,  $f = -2$ ,  $c = 25$ .

Hence, the circle (i) becomes

$$x^2 + y^2 - 22x - 4y + 25 = 0 \quad \dots \quad \dots \quad \text{(ii)}$$

The given straight line is

$$3x - 4y + 5 = 0 \quad \dots \quad \dots \quad \dots \quad \text{(iii)}$$

For the common points of intersection of (ii) and (iii), eliminating  $y$ , the abscissæ are the roots of

$$x^2 + \left(\frac{3x+5}{4}\right)^2 - 22x - (8x+5) + 25 = 0,$$

$$\text{or, } 5x^2 - 74x + 69 = 0 \quad \dots \quad \dots \quad \dots \quad \text{(iv)}$$

If  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the points of intersection of (ii) and (iii), then  $x_1$  and  $x_2$  are roots of (iv).

$$\therefore x_1 + x_2 = \frac{7}{5}, x_1 x_2 = -\frac{9}{5}.$$

$$\therefore (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2 = \left(\frac{7}{5}\right)^2 - 4 \cdot \frac{-9}{5} = \frac{49}{25} + \frac{36}{5} = \frac{205}{25}.$$

Also,  $(x_1, y_1)$  and  $(x_2, y_2)$  both lying on (iii),

$$3x_1 - 4y_1 + 5 = 0, \quad 3x_2 - 4y_2 + 5 = 0.$$

$$\therefore 3(x_1 - x_2) - 4(y_1 - y_2) = 0. \quad \therefore (y_1 - y_2)^2 = \frac{9}{16}(x_1 - x_2)^2.$$

$\therefore$  the length of the intercepted chord being  $l$ ,

$$\begin{aligned} l^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 = \left(1 + \frac{9}{16}\right)(x_1 - x_2)^2 \\ &= \frac{25}{16} \times \frac{205}{25} = 256. \end{aligned}$$

$$\therefore l = 16.$$

*Alternatively*

The centre of the circle (ii) is  $(11, 2)$  and its radius  $r$  is  $\sqrt{11^2 + 2^2 - 25} = 10$   
[ See § 4'3 ].

Perpendicular from this centre on line (iii) is

$$p = \frac{3 \cdot 11 - 4 \cdot 2 + 5}{\sqrt{3^2 + 4^2}} = 6.$$

Now if  $AB$  be the chord along the line, and  $CN$  be the perpendicular on  $AB$  from the centre  $C$ , then  $N$  is the mid-point of  $AB$ . Also  $AN^2 = CA^2 - CN^2$ .

Hence, the length of the chord intercepted is

$$AB = 2AN = 2\sqrt{r^2 - p^2} = 2\sqrt{100 - 36} = 16.$$

**Ex. 3.** Show that the straight line  $4x + 3y - 31 = 0$  touches the circle  $x^2 + y^2 - 6x + 4y = 12$ , and find the point of contact.

If possible, let  $(x_1, y_1)$  be the co-ordinates of the point on the circle

$$x^2 + y^2 - 6x + 4y - 12 = 0 \quad \dots \quad (i)$$

at which the given line

$$4x + 3y - 31 = 0 \quad \dots \quad (ii)$$

is a tangent.

As the equation to the tangent at  $(x_1, y_1)$  of the circle (i) is

$$xx_1 + yy_1 - 3(x + x_1) + 2(y + y_1) - 12 = 0,$$



Hence, if  $(x_1, y_1)$  and  $(x_2, y_2)$  be the extremities of the chord of intersection of (i) with (ii),  $x_1, x_2$  being roots of the above equation.

$$x_1 + x_2 = -\frac{2mc}{1+m^2}.$$

Thus,  $X, Y$  being the middle point of the chord

$$X = \frac{1}{2}(x_1 + x_2) = -\frac{mc}{1+m^2}.$$

Also,  $X, Y$  being a point on (ii),  $Y = mX + c$ .

Eliminating  $c$  between these,

$$X = -\frac{m}{1+m^2} (Y - mX), \text{ or, } X + mY = 0.$$

As this is free from  $c$ , this equation is satisfied by the co-ordinates of the middle point of every chord of the system, and it evidently represents the equation to a straight line passing through the origin i.e., passing through the centre of the circle, and is thus a diameter.

### Examples IV

1. Obtain the equation to a circle having its centre at (3, 7) and diameter 10.

What is the length of the intercept of this circle on the  $y$ -axis?  
[ H. S. 1960, *Compartmental* ]

2. The extremities of a diameter of a circle have co-ordinates  $(-4, 3)$  and  $(12, -1)$ . Find the equation to the circle. What length does it intercept on the  $y$ -axis?

[ H. S. 1961, *Compartmental* ]

3. Show that the equation  $3x^2 + 3y^2 - 5x - 6y + 4 = 0$  represents a circle, and find its radius and co-ordinates of its centre.

4. Find the equation of the circle passing through the points  $(0, 1)$ ,  $(0, -1)$ ,  $(1, 0)$ , and determine its centre and radius.  
[ H. S. 1961 ]

5. Obtain the co-ordinates of the centre of the circle passing through the points  $(1, 2)$ ,  $(3, -4)$ ,  $(5, -6)$ , and determine the length of its diameter.

Is the origin inside, or outside the circle? [ H. S. 1960 ]

[ Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since  $(1, 2)$ ;  $(3, -4)$  and  $(5, -6)$  lie on circle, hence they satisfy the above equation.

$$\therefore 2g + 4f + c + 5 = 0$$

$$6g - 8f + c + 25 = 0$$

$$10g - 12f + c + 61 = 0.$$

Solving the equation,  $g = -11$ ;  $f = -2$ ;  $c = 25$ .

Co-ordinates of centre  $(-g, -f)$  i.e.,  $(11, 2)$ .

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{(-11)^2 + (-2)^2 - 25} = 10.$$

$$\therefore \text{Diameter} = 2 \times 10 = 20.$$

Length from origin to centre

$$= \sqrt{(-11)^2 + (-2)^2} = \sqrt{125} = 5\sqrt{5} > 10 \text{ (radius of circle.)}$$

Hence origin lies outside the circle. ]

6. Find the equation to a circle which passes through the points  $(0, -3)$  and  $(3, -4)$ , and which has its centre on the straight line  $2x - 5y + 12 = 0$ .

7. Find the equation to the circle passing through the origin and having intercepts 4 and  $-6$  on the  $x$ -axis and  $y$ -axis respectively.

8. Find the equations to the circles which touch the axis of  $x$  and pass through the points  $(1, -2)$  and  $(3, -4)$ .

9.  $A$  and  $B$  are two fixed points on a plane and the point  $P$  moves on the plane in such a way that  $PA = 2PB$  always. Prove analytically that the locus of  $P$  is a circle.

[ H. S. 1961, Compartmental ]

[  $x$ -axis is taken along  $AB$  with the origin at mid-point of  $AB$  and let  $AB = 2a$ . So that the co-ordinates of  $A$  and  $B$  are  $(-a, 0)$  and  $(a, 0)$ . And let the coordinates of  $P$  be  $(x_1, y_1)$ .

Hence from the given condition

$$\sqrt{(x_1 + a)^2 + (y_1 - 0)^2} = 2 \sqrt{(x_1 - a)^2 + (y_1 - 0)^2}.$$

$$\text{or } (x_1 + a)^2 + y_1^2 = 4 [(x_1 - a)^2 + y_1^2]$$

$$\text{or } x_1^2 + y_1^2 - \frac{1}{3} 2ax_1 + \frac{4}{3}a^2 = 0.$$

Hence locus of  $P$  is  $x^2 + y^2 - \frac{10}{3}ax + \frac{4}{3}a^2 = 0$  which is the equation of the circle with the centre as  $(\frac{5}{3}a, 0)$  and

$$r = \sqrt{\left(\frac{5a}{3}\right)^2 - \frac{4}{3}a^2} = \frac{\sqrt{13}}{3}a.]$$

10.  $B, C$  are fixed points having co-ordinates  $(3, 0)$  and  $(-3, 0)$  respectively. If the vertical angle  $BAC$  be  $90^\circ$ , show that the locus of the centroid of the triangle  $ABC$  is a circle whose equation you are to determine. [H. S. 1961]

11. (i) Find the length of the chord of the circle  $x^2 + y^2 = 64$ , intercepted on the straight line  $3x + 4y - c = 0$ .

[As in § 4.12,  $c \times -2$  for the common point of intersections eliminating  $y$  between  $x^2 + y^2 = 64$  and  $3x + 4y - c = 0$ , the abscissae are the roots of

$$x^2 + \left(\frac{c-3x}{4}\right)^2 = 64$$

$$\text{or } 25x^2 - 6cx + c^2 - 64 \times 16 = 0.$$

If  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the point of intersection, then  $x_1, x_2$  are the roots of the above equation.

$$\therefore x_1 + x_2 = \frac{6c}{25} \text{ and } x_1 x_2 = \frac{c^2 - 64 \times 16}{25}$$

$$\begin{aligned} \therefore (x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1 x_2 \\ &= \frac{36c^2}{625} - \frac{4c^2 - 64 \times 64}{25} = \frac{64 \times 64 \times 25 - 625(c^2 - 4096)}{625} \end{aligned}$$

Also  $(x_1, y_1)$  and  $(x_2, y_2)$  both on  $3x + 4y - c = 0$ ,

$$\therefore 3y_1 + 4y_2 - c = 0 \text{ and } 3x_2 + 4y_2 - c = 0$$

$$\text{or } 3(x_1 - x_2) = -4(y_1 - y_2). \therefore (y_1 - y_2)^2 = \frac{9}{16}(x_1 - x_2)^2.$$

Now if length of the intercepted chord be ' $l$ ',

$$\begin{aligned} \therefore l^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 = \frac{25}{16}(x_1 - x_2)^2 \\ &= \frac{25}{16} \cdot \frac{64 \times 64 \times 25 - 625(c^2 - 4096)}{625} = \frac{64 \times 100 - 4c^2}{25} \\ &= \frac{4}{25}(1600 - c^2). \therefore l = \frac{2}{5}\sqrt{1600 - c^2}.] \end{aligned}$$

(ii) Obtain the co-ordinates of the points of contact of any one of the two tangents to the above circle  $x^2 + y^2 = 64$ , parallel to the line  $3x + 4y - c = 0$ . [H. S. 1960]

12. Prove that the straight line  $y = x + a\sqrt{2}$  touches the circle  $x^2 + y^2 = a^2$ , and find its point of contact. [H. S. 1961]

13. Show that the line  $3x + 4y + 7 = 0$  touches the circle  $x^2 + y^2 - 4x - 6y - 12 = 0$ , and find its point of contact.

14. Determine whether the straight line  $x + y = 2 + \sqrt{2}$  touches the circle  $x^2 + y^2 - 2x - 2y + 1 = 0$ . If it does, find the co-ordinates of the point of contact.

15. Find the equation to the circle

(i) having its centre at the point (3, 4), and touching the straight line  $5x + 12y + 2 = 0$ .

(ii) having its centre at (1, -3) and touching the straight line  $2x - y - 4 = 0$ .

16. Find the points at which the tangents to the circle  $x^2 + y^2 - 6x + 8y = 0$  is parallel to the line  $3x + 4y = 0$ .

17. Find the points on the circle  $x^2 + y^2 - 2x + 6y - 58 = 0$  at which the tangents are perpendicular to the line  $4x - y = 2$ .

[Equation of any line perpendicular to

$$4x - y = 2 \text{ is } x + 4y = c \quad \dots \quad (1)$$

Eliminating  $x$  between (1) and  $x^2 + y^2 - 2x + 6y - 58 = 0$  ... (2)

we get  $(c - 4y)^2 + y^2 - 2(c - 4y) + 6y - 58 = 0$

$$\text{or, } 17y^2 + 2y(7 - 4c) + (c^2 - 2c - 58) = 0.$$

If (1) touches (2), the two roots of above equations must be equal.

$$\therefore 4(7 - 4c)^2 = 4 \cdot 17 \cdot (c^2 - 2c - 58) \text{ whence } c = 23, -45.$$

Let  $(x_1, y_1)$  be the point of contact of tangent. Equation of tangent at  $(x_1, y_1)$  of the circle (2) is

$$xx_1 + yy_1 - (x + x_1) + 3(y + y_1) - 58 = 0$$

$$\text{or } x(x_1 - 1) + y(y_1 + 3) - (x_1 - 3y_1 + 58) = 0. \quad \dots (3)$$

If (1) touches (2), (1) and (3) must be identical,

$$\therefore \frac{x_1 - 1}{1} = \frac{y_1 + 3}{4} = \frac{x_1 - 3y_1 + 58}{c}$$

$\therefore$  solving and substituting the values of  $c$ ,

$$x_1 = 3, y_1 = 5 \text{ and } x_1 = -1, y_1 = -1.$$

Hence, two points of contact are  $(-1, -1)$  and  $(3, 5)$ . ]

## CIRCLE

**18.** Show that the two circles

$$(i) \ x^2 + y^2 + 6x + 14y + 9 = 0 \quad \text{and}$$

$$x^2 + y^2 - 4x - 10y - 7 = 0$$

touch each other externally.

[ Co-ordinates of the circle are  $(-3, -7)$  and radius  $= \sqrt{(3)^2 + (7)^2 - 9} = 7 = r_1$   
of the circle  $x^2 + y^2 + 6x + 14y + 9 = 0$ .

For the second circle  $x^2 + y^2 - 4x - 10y - 7 = 0$ , co-ordinates of the centre  
 $(2, 5)$ ,  $r_2 = \sqrt{(-2)^2 + (-5)^2 - 7} = 6$ .

$$\therefore r_1 + r_2 = 13.$$

Distance between centres of two circles

$$= \sqrt{(-3-2)^2 + (-7-5)^2} = 13 = r_1 + r_2.$$

Hence two circles touch each other externally. ]

$$(ii) \ x^2 + y^2 - 6x + 6y - 18 = 0 \quad \text{and}$$

$$x^2 + y^2 - 2y = 0$$

touch each other internally.

**19.** Find the length of the tangent drawn from

(i) the point  $(-3, 11)$  to the circle

$$x^2 + y^2 - 4x + 2y - 20 = 0.$$

[ As in § 4.11, let  $bc$  be the length of tangent,

$$(i) \ b = \sqrt{(-3)^2 + (11)^2 - 4(-3) + 2 \cdot 11 - 20} = \sqrt{144} = 12. ]$$

(ii) the point  $(7, 2)$  to the circle

$$2x^2 + 2y^2 + 5x + y - 15 = 0.$$

**20.** Show that the locus of the points from which the lengths of the tangents to the circles

$$x^2 + y^2 - 3x + 4y - 7 = 0$$

$$\text{and} \quad x^2 + y^2 + 2x - 5y + 1 = 0$$

are equal, is a straight line perpendicular to the line joining the centres of the circles.

[ Let  $(x_1, y_1)$  be a point such that lengths from this to two given circles are equal.



$$\text{Hence, } \sqrt{x_1^2 + y_1^2 - 3x_1 + 4y_1 - 7} = \sqrt{x_1^2 + y_1^2 + 2x_1 - 5y_1 + 1}$$

$$\text{or } x_1^2 + y_1^2 - 3x_1 + 4y_1 - 7 = x_1^2 + y_1^2 + 2x_1 - 5y_1 + 1$$

$$\text{or } 5x_1 - 9y_1 + 8 = 0. \quad \dots \quad (1)$$

Hence locus of  $(x_1, y_1)$  is  $5x - 9y + 8 = 0$  which is a straight line and 'm' of this line say  $m_1 = \frac{5}{9}$ .

Centre of  $x^2 + y^2 - 3x + 4y - 7 = 0$  is  $(\frac{3}{2}, -2)$ .

Centre of  $x^2 + y^2 + 2x - 5y + 1 = 0$  is  $(-1, \frac{5}{2})$ .

Equation of straight line passing through  $(\frac{3}{2}, -2)$  and  $(-1, \frac{5}{2})$  is

$$\frac{x - \frac{3}{2}}{-1 - \frac{3}{2}} = \frac{y + 2}{\frac{5}{2} + 2} \quad \text{or} \quad \frac{x - \frac{3}{2}}{-\frac{5}{2}} = \frac{y + 2}{\frac{9}{2}} \quad \dots \quad (2)$$

'm' of this line say,  $m_2 = -\frac{9}{5}$ .

Since,  $m_1 m_2 = \frac{5}{9} \times -\frac{9}{5} = -1$ , hence (1) and (2) are perpendicular. ]

### ANSWERS

1.  $x^2 + y^2 - 6x - 14y + 33 = 0$ ; 8.
2.  $x^2 + y^2 - 8x - 2y - 51 = 0$ ;  $4\sqrt{13}$ .
3.  $\frac{1}{2}\sqrt{13}$ ;  $(\frac{5}{2}, 1)$ .
4.  $x^2 + y^2 - 6x + 2y - 15 = 0$ ;  $(3, -1)$ ; 5.
5.  $(11, 2)$ ; 20; outside.
6.  $x^2 + y^2 - 8x - 8y - 33 = 0$ .
7.  $x^2 + y^2 - 4x + 6y = 0$ .
8.  $x^2 + y^2 - 6x + 4y + 9 = 0$ ,  $x^2 + y^2 + 10x + 20y + 25 = 0$ .
10.  $x^2 + y^2 = 1$ .
11. (i)  $\frac{2}{5}\sqrt{1600 - c^2}$ . (ii)  $(\frac{2}{5}^4, \frac{2}{5}^2)$  or  $(-\frac{2}{5}^4, -\frac{2}{5}^2)$ .
12.  $(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$ .
13.  $(-1, -1)$ .
14. Yes;  $(1 + \frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}})$ .
15. (i)  $x^2 + y^2 - 6x - 8y = 0$ .
- (ii)  $5x^2 + 5y^2 - 10x + 30y + 49 = 0$ .
16.  $(6, 0)$  and  $(0, -8)$ .
17.  $(3, 5)$  and  $(-1, -11)$ .
19. (i) 12.
- (ii) 8.

## CHAPTER V

### CONICS

#### 5'1. Definitions.

If a point moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the locus traced out by the point is defined to be a **conic**.

The fixed point is called the **focus** of the conic, and is usually denoted by the letter  $S$ .

The fixed straight line is referred to as the **directrix** of the conic.

The straight line through the focus perpendicular to the directrix is called the **axis**.

The constant ratio (of the distance of any point on a conic from the focus to the perpendicular distance of the point from the directrix) is called the **eccentricity** of the conic, and is usually denoted by the letter  $e$ .

When  $e = 1$ , the conic is defined to be a **parabola**.

When  $e < 1$ , the conic is defined to be an **ellipse**.

When  $e > 1$ , the conic is defined to be a **hyperbola**.

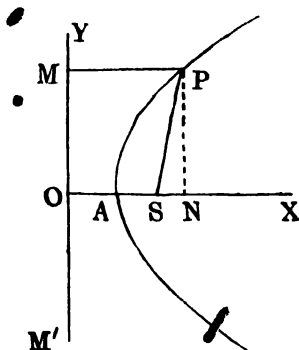
**Note.** The name 'Conic' or 'Conic-section' is due to the fact that these curves were first obtained and studied as sections of a cone by planes in various ways.

#### 5'2. Parabola.

(A) *Equation with axis and directrix as axes of co-ordinates.*

Let  $S$  be the focus, and  $MM'$  the directrix of the parabola, which are fixed, and let  $OSX$  be the straight line through  $S$  perpendicular to the directrix, so that it is the axis of the parabola,  $O$  being its point of intersection with the directrix.

Let us take  $OX$  as  $x$ -axis and  $OY$  (along the directrix) as  $y$ -axis, and let  $(x, y)$  be the co-ordinates of any point  $P$  on the



parabola. If  $PN$  and  $PM$  be perpendiculars from  $P$  on  $OX$  and  $OY$ , then  $PM = ON = x$ ,  $PN = y$ .

Let the distance  $OS$  of  $S$  from the directrix be  $d$ . Then co-ordinates of  $S$  are  $(d, 0)$ .

From the definition of the parabola,

$$\frac{PS}{PM} = 1, \text{ or, } PS = PM. \quad PS^2 = PM^2$$

$$\text{or, } (x - d)^2 + y^2 = x^2, \quad \therefore y^2 = 2d(x - \frac{1}{2}d)$$

or writing  $d = 2a$ , this can be written as

$$y^2 = 4a(x - a) \quad \dots \quad \dots \quad (i)$$

If  $A$  be the middle point of  $OS$ , clearly  $OA = AS = a$ .

The co-ordinates of  $A$  are then  $(a, 0)$  and they evidently satisfy the equation (i). Thus,  $A$  is a point on the parabola. This point  $A$  is called the **vertex** of the parabola.

**(B) Standard form of the equation to a parabola.**

If we transfer the origin to the vertex  $A$ , the equation (i) of the parabola reduces to

$$y^2 = 4ax \quad \dots \quad \dots \quad (ii)$$

which is the standard form of the equation to a parabola.

Here the vertex is the origin, the axis of the parabola is the  $x$ -axis, and the line through the vertex parallel to the directrix is the  $y$ -axis,  $a$  being the distance of the vertex from the focus, which is also equal to the perpendicular distance of the vertex from the directrix.

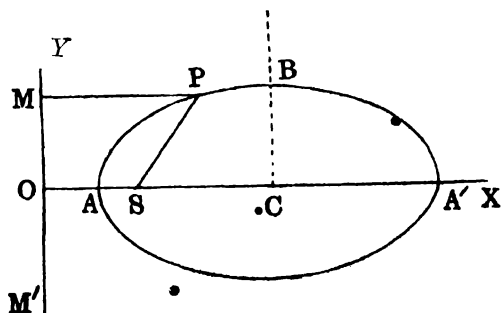
**Note.** For a discussion on the shape of the parabola and its elementary properties, see Chapter VI. •

### 5.3. Ellipse.

(A) *Equation with directrix as  $y$ -axis and perpendicular to it through the focus as  $x$ -axis.*

Let  $S$  be the focus,  $MM'$  the directrix, and  $e (< 1)$  be the eccentricity of the ellipse.

$SO$  being perpendicular on  $MM'$ ,  $OSX$  is taken as the  $x$ -axis, and  $OY$  (along the directrix) as  $y$ -axis. Let  $(x, y)$  denote the co-ordinates of any point  $P$  on the ellipse. Let the distance  $SO$



(of focus from the directrix) be  $d$ , and let  $PM$  be the perpendicular from  $P$  on the directrix, so that  $PM = x$ .

Now, from the definition of the ellipse,

$$\frac{PS}{PM} = e, \quad \text{or,} \quad PS = e \cdot PM.$$

$$\therefore PS^2 = e^2 \cdot PM^2.$$

Hence, co-ordinates of  $S$  being evidently  $(d, 0)$ ,

$$(x - d)^2 + y^2 = e^2 x^2 \quad (i)$$

This is then the equation to the ellipse with directrix as  $y$ -axis, and perpendicular to it through the focus as  $x$ -axis,  $d$  being the distance of the focus from the directrix.

(B) *Standard form of the equation to an ellipse.*

The above equation (i) can be written in the form

$$x^2(1-e^2) - 2dx + d^2 + y^2 = 0,$$

$$\text{or, } (1-e^2)\left(x - \frac{d}{1-e^2}\right)^2 + y^2 = \frac{d^2}{1-e^2} - d^2 = \frac{d^2 e^2}{1-e^2},$$

$$\text{or, } \left(x - \frac{d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \left(\frac{de}{1-e^2}\right)^2.$$

Writing  $\frac{de}{1-e^2} = a$ , and transferring the origin to  $C$  whose co-ordinates are  $\left(\frac{d}{1-e^2}, 0\right)$  i.e.,  $\left(a, 0\right)$ , (the axes remaining parallel to their original directions), the equation to the ellipse reduces to its standard form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1,$$

$$\text{i.e., } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots \quad \dots \quad \text{(ii)}$$

where  $b^2 = a^2(1-e^2)$ .

Here the origin  $C$  is on the axis perpendicular to the directrix through the focus, at a distance  $\frac{d}{1-e^2} = \frac{a}{e}$  from the directrix, and the  $y$ -axis,  $CB$  is parallel to the directrix.

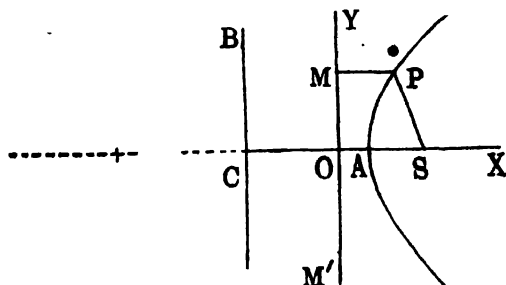
The point  $C$  is called the **centre** of the ellipse, the reason for which will be explained in Chapter VII.

$$\text{Distance, } CS = \frac{d}{1-e^2} - d = \frac{e^2 d}{1-e^2} = ae.$$

**Note.** For a discussion on the shape of the ellipse, and its elementary properties, see Chapter VII.

### 5.4. Hyperbola.

(A) *Equation with directrix as  $y$ -axis, and perpendicular to it through the focus as  $x$ -axis.*



Let  $S$  be the focus,  $MM'$  the directrix, and  $e (> 1)$  be the eccentricity of the hyperbola.

$SO$  being perpendicular on  $MM'$ ,  $OSX$  is taken as the  $x$ -axis, and  $OY$  (along the directrix) as  $y$ -axis. Let  $(x, y)$  denote the co-ordinates of any point  $P$  on the hyperbola. Let the distance  $SO$  (of focus from the directrix) be  $d$ , and let  $PM$  be the perpendicular from  $P$  on the directrix, so that  $PM = x$ .

Now, from the definition of the hyperbola,

$$\frac{PS}{PM} = e, \text{ or, } PS = e \cdot PM.$$

$$\therefore PS^2 = e^2 \cdot PM^2.$$

Hence, co-ordinates of  $S$  being evidently  $(d, 0)$ ,

$$(x - d)^2 + y^2 = e^2 x^2 \quad \dots \quad (i)$$

This is then the equation to the hyperbola with directrix as  $y$ -axis, and perpendicular to it through the focus as  $x$ -axis,  $d$  being the distance of the focus from the directrix.

(B) *Standard form of the equation to a hyperbola.*

The above equation (i) can be written in the form ( $e$  being greater than 1 here);

$$x^2(e^2 - 1) + 2dx - y^2 = d^2;$$

$$\text{or, } (e^2 - 1) \left( x + \frac{d}{e^2 - 1} \right)^2 - y^2 = d^2 + \frac{d^2}{e^2 - 1} - \frac{e^2 d^2}{-1}$$

$$\text{or, } \left( x + \frac{d}{e^2 - 1} \right)^2 - \frac{y^2}{e^2 - 1} = \left( \frac{de}{e^2 - 1} \right)^2.$$

Writing  $\frac{de}{e^2 - 1} = a$ , and transferring the origin to  $C$  whose co-ordinates are  $\left( -\frac{d}{e^2 - 1}, 0 \right)$  i.e.,  $\left( -\frac{a}{e}, 0 \right)$ , (the axes remaining parallel to their original directions) the equation to the hyperbola reduces to its standard form

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1,$$

$$\text{or, } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where  $b^2 = a^2(e^2 - 1)$ .

Here the origin  $C$  is on the axis perpendicular through the focus to the directrix, at a distance  $\frac{d}{e^2 - 1} = \frac{a}{e}$  from the directrix on the side opposite to the focus. This point  $C$  is called the *centre* of the hyperbola.

$$\text{Distance, } CS = d + \frac{d}{e^2 - 1} = \frac{de^2}{e^2 - 1} = ae.$$

**Note.** For a discussion on the shape of the hyperbola, as also its elementary properties, see Chapter VIII.

### 5.5. Examples.

**Ex. 1.** Find out the equation to the parabola whose focus is  $(-3, 4)$  and directrix is  $6x - 7y + 5 = 0$ . [H. S. 1961]

Let  $(x_1, y_1)$  be the co-ordinates of any point on the parabola.

Its distance from the given focus  $(-3, 4)$  is  $\sqrt{(x_1 + 3)^2 + (y_1 - 4)^2}$ , and its perpendicular distance from the given directrix  $6x - 7y + 5 = 0$  is  $\frac{6x_1 - 7y_1 + 5}{\sqrt{6^2 + 7^2}}$ .  
For the parabola these two distances are equal.

## CONICS

Hence,  $(x_1+3)^2 + (y_1-4)^2 = \frac{(6x_1-7y_1+5)^2}{8^2+7^2}$ . Thus, the co-ordinates  $(x_1, y_1)$  of any point on the parabola satisfy the equation

$$85\{(x+3)^2 + (y-4)^2\} = (6x-7y+5)^2,$$

$$\text{or, } 49x^2 + 84xy + 36y^2 + 450x - 610y + 2100 = 0$$

which is then the required equation to the parabola. •

**Ex. 2.** Find the equation to the ellipse, whose focus is the point  $(-1, 1)$ , and directrix is the line  $x-y+3=0$ , and whose eccentricity is  $\frac{1}{2}$ .

Let  $(x_1, y_1)$  be the co-ordinates of any point on the ellipse.

Its distance from the given focus  $(-1, 1)$  is  $\sqrt{(x_1+1)^2 + (y_1-1)^2}$ , and its perpendicular distance from the given directrix  $x-y+3=0$  is  $\frac{x_1-y_1+3}{\sqrt{1+1}}$ . The ratio of these two distances is equal to the given eccentricity  $\frac{1}{2}$  for any point on the ellipse.

$$\text{Hence, } \sqrt{(x_1+1)^2 + (y_1-1)^2} = \frac{1}{2} \frac{x_1-y_1+3}{\sqrt{2}},$$

$$\text{or, } 8\{(x_1+1)^2 + (y_1-1)^2\} = (x_1-y_1+3)^2.$$

Thus, the co-ordinates  $(x_1, y_1)$  of the any point on the ellipse satisfy the equation

$$8\{(x+1)^2 + (y-1)^2\} = (x-y+3)^2,$$

$$\text{or, } 7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0,$$

which is then the required equation of the ellipse in question. • •



## CHAPTER VI

### PARABOLA

#### 6'1. Parabola.

As has been defined in the previous chapter, a *parabola* is a curve traced out by a point which moves on a plane such that its distance from a fixed point on the plane is always equal to its perpendicular distance from a fixed straight line on that plane.

The fixed point is called the *focus*, and the fixed straight line is called the *directrix*.

#### 6'2. Standard equation of a parabola.

Let  $S$  be the focus, and  $MM'$  the directrix of the parabola. Draw  $SZ$  perpendicular from  $S$  on  $MM'$ , and let  $A$  be the middle point of  $ZS$ . Clearly,  $\therefore AS = AZ$ , the point  $A$  is a point on the parabola. This point  $A$  is called the *vertex* of the parabola.

Let  $AS$  (the distance of the vertex from the focus) =  $a$ . Then  $AZ$  is also =  $a$ , and  $SZ = 2a$ .

Take  $A$  as origin, and  $ASX$  (perpendicular to the directrix through  $S$ ) as  $x$ -axis,  $AY$  parallel to the directrix through  $A$  being the  $y$ -axis. Clearly the co-ordinates of  $S$  are  $(a, 0)$ .

$P$  being any point on the parabola whose co-ordinates are  $(x, y)$ , if  $PN$  be perpendicular on  $AX$ , and  $PM$  perpendicular to the directrix,

then  $PM = ZN = AZ + AN = a + x$ .

Now, from the definition of a parabola,

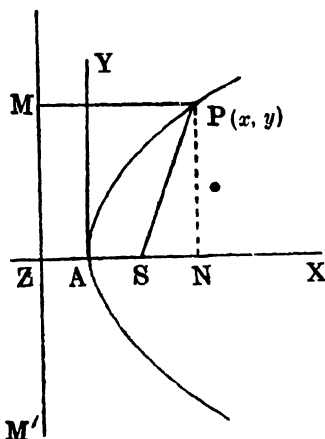
$$PS = PM, \text{ or, } PS^2 = PM^2.$$

$$\therefore (x - a)^2 + y^2 = (a + x)^2.$$

$$\therefore y^2 = 4ax.$$

...

... (i)



This being the relation satisfied by the co-ordinates of any point on the parabola, it represents the equation to the parabola in standard form with vertex as origin.

Here  $a$  represents the distance of the vertex from the focus (or from the directrix, the two being equal)<sup>1)</sup>

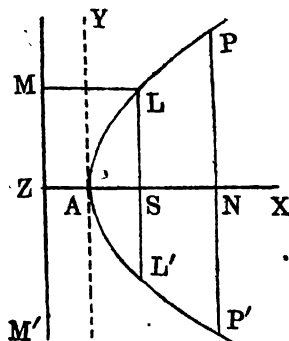
The line  $AX$  (perpendicular to the directrix through  $S$ ), which is chosen as the axis of  $x$  here, is referred to as the *axis* of the parabola.

**Note.** In the previous chapter the equation was obtained with  $Z$  as origin, and then by transference of the origin to  $A$ , the equation to the parabola was obtained in the above standard form.

### 6.3. Shape and elementary properties of the parabola.

From the equation  $y^2 = 4ax$ , it is evident that if  $x$  be negative,  $y^2$  being negative,  $y$  is imaginary. Hence, there is no real point with  $x$  negative, that is, there is no point of the parabola to the left of the origin  $A$ .

Again, for every positive value of  $x$ , there are two equal and opposite values of  $y$ . Hence corresponding to a point  $P$  on the parabola with positive ordinate  $PN$ , there is a point  $P'$  on the parabola with the same  $x$  ( $=AN$ ) with equal negative ordinate  $P'N$ . In other words, the parabola is symmetrical with respect to the  $x$ -axis  $AX$ , which bisects every chord of the type  $PNP'$  perpendicular to it. As  $x$  diminishes, and ultimately becomes zero, the two ordinates, which are equal in value and opposite in sign, become zero, and the point coincides with the origin  $A$ , which is the vertex of the parabola. As  $x$  becomes larger and larger, the values of  $y$  also become larger in magnitude. Hence, the shape of the parabola is as shown in the figure, closed at the left end  $A$ , and open on the right,  $y$  gradually becoming numerically larger and larger with  $x$ , the whole curve being symmetrical about  $OX$ .



It is for this property that  $OX$  is defined as the axis, and  $A$  is called the vertex.

A chord  $PNP'$  perpendicular to the axis (i.e., parallel to the directrix) and bisected by the axis, is called a *double ordinate*.  $PN$  or  $P'N$  is the ordinate of  $P$  or  $P'$ .

The chord  $LSL'$  through the focus  $S$  parallel to the directrix (and so perpendicular to the axis) is called the **Latus Rectum**.

If  $LM$  be the perpendicular on the directrix from the extremity  $L$  of the latus rectum, from the property of the parabola,  $LS = LM = ZS = 2AS = 2a$ .

Thus, the **Latus rectum** =  $4a$ ,

i.e., the *latus rectum* is four times the distance of the focus from the vertex, or, double the distance of the focus from the directrix.

The equation  $y^2 = 4ax$  asserts the geometrical property of the parabola, that  $PN^2 = 4AS \cdot AN$ , or, *the square on the ordinate is equal to the rectangle contained by the abscissa and the latus rectum*.

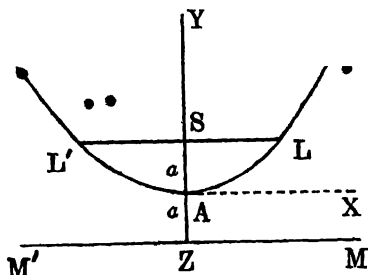
To sum up, we note that for the standard equation  $y^2 = 4ax$  of the parabola,

- (i) the vertex is the origin ;
  - (ii) the length of the latus rectum is  $4a$  ;
  - (iii) the focus has co-ordinates  $(a, 0)$  ;
  - (iv) the directrix has equation  $x = -a$  ;
  - (v) the axis is the axis of  $x$  ;
- and (vi) the co-ordinates of the extremities of the latus rectum are  $(a, 2a)$  for  $L$ , and  $(a, -2a)$  for  $L'$ .

**Note.** Equations  $x^2 = 4ay$ ,  $y^2 = -4ax$ ,  $x^2 = -4ay$ .

If in the equation  $y^2 = 4ax$  of a parabola, the axes of  $x$  and  $y$  are interchanged, i.e., choosing the vertex  $A$  as origin, and latus rectum being  $4a$  as before, the axis of the parabola (the line perpendicular to the directrix through the focus) be taken along the  $y$ -axis, the  $x$ -axis being parallel to the directrix, the equation to the parabola becomes  $x^2 = 4ay$  and the figure is as shown here.

Here, the co-ordinates of the focus are  $(0, a)$  and equation to the directrix is  $y = -a$ . The co-ordinates of the extremities  $L$  and  $L'$  of the latus rectum are  $(2a, a)$  and  $(-2a, a)$  respectively.

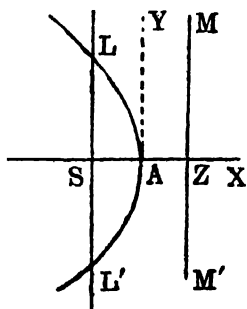


$$x^2 = 4ay$$

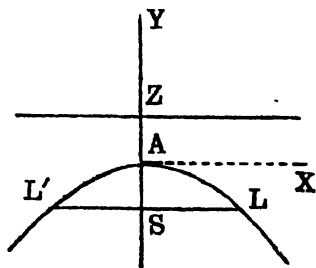
If in the equation  $y^2 = 4ax$ , the direction of the  $x$ -axis is reversed, i.e., if the positive direction of the  $x$ -axis be taken from the vertex towards the directrix (the direction from the vertex to the focus being negative), the equation becomes  $y^2 = -4ax$ , and the figure is as shown below, the concavity of the parabola being towards the negative side of the  $x$ -axis.

The co-ordinates of the focus are  $(-a, 0)$  and directrix is  $x = a$ .

Similarly, in the equation  $x^2 = 4ay$ , if the direction of  $y$ -axis is reversed, the equation becomes  $x^2 = -4ay$ , and the figure is as shown below, the concavity of the parabola being towards the negative side of the  $y$ -axis.



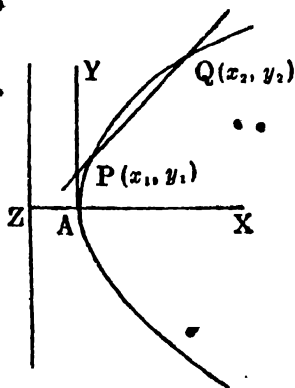
$$y^2 = -4ax$$



$$x^2 = -4ay$$

The co-ordinates of the focus are  $(0, -a)$  and directrix is  $y = a$ .

**6'4. Equation to the tangent at a given point  $(x_1, y_1)$  on the parabola  $y^2 = 4ax$ .**



Let  $P$  be the point  $(x_1, y_1)$  on the parabola  $y^2 = 4ax$  (i)

and let  $Q(x_2, y_2)$  be a neighbouring point on it.

The equation to the chord  $PQ$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad \dots \quad \text{(ii)}$$

Now since  $P$  and  $Q$  both lie on the parabola (i),

we have  $y_1^2 = 4ax_1$  ... (iii) and  $y_2^2 = 4ax_2$  (iv)

$\therefore$  substituting,  $y_2^2 - y_1^2 = 4a(x_2 - x_1)$ ,

$$\text{or, } \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1} \quad \bullet$$

equation (ii) can be written as

$$y - y_1 = \frac{4a}{y_2 + y_1} (x - x_1) \quad \dots \quad \text{(v)}$$

Now make  $Q$  approach  $P$  and ultimately coincide with it, so that the co-ordinates  $(x_2, y_2)$  coincide with  $(x_1, y_1)$ . In that limiting position, the straight line  $PQ$  becomes the tangent at  $P$ , whose equation [ from (v) ] then becomes

$$y - y_1 = \frac{4a}{2y_1} (x - x_1), \quad \text{or, } yy_1 - y_1^2 = 2a(x - x_1),$$

$$\text{i.e., } yy_1 = y_1^2 + 2a(x - x_1) = 4ax_1 + 2a(x - x_1) \quad [\text{by (iii)}].$$

Hence, the equation to the tangent at  $(x_1, y_1)$  is

$$yy_1 = 2a(x + x_1).$$

Cor. The tangent at the vertex of the parabola  $y^2 = 4ax$  is the  $y$ -axis.

**6'5. Equation to the normal at  $(x_1, y_1)$  to the parabola  $y^2 = 4ax$ .**

The tangent at  $(x_1, y_1)$  to the parabola  $y^2 = 4ax$  is

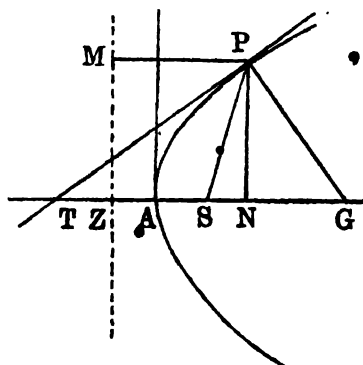
$$yy_1 = 2a(x + x_1), \text{ or, } y = \frac{2a}{y_1}(x + x_1),$$

of which the 'm' is  $\frac{2a}{y_1}$ .

The normal, which is perpendicular to the tangent through  $(x_1, y_1)$  is then

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

**6'6. Tangent and normal properties : Subtangent and Subnormal.**



The length of the axis intercepted between the tangent and the foot of the ordinate of any point on the parabola is defined as the *subtangent* of the point.

The length of the axis intercepted between the normal and the foot of the ordinate of any point on the parabola is defined as the *subnormal* of the point.

Thus, if  $PT$  and  $PG$  be the tangent and normal at  $P$ , intersecting the axis at  $T$  and  $G$  respectively, and  $PN$  be the ordinate of  $P$ , then

$TN$  is the subtangent of  $P$  and  $NG$  is the subnormal.

Now from the equation  $yy_1 = 2a(x+x_1)$  of the tangent at  $P(x_1, y_1)$ , the point  $T$  where it intersects the axis is obtained by putting  $y=0$ , and thus for  $T$ ,  $x+x_1=0$  i.e.,  $x=-x_1$ .

Hence,  $AT=AN$  in magnitude,  $T$  being on the negative side of  $A$ .

Hence, we get the geometrical property of the parabola that *the subtangent of any point on a parabola is bisected at the vertex.*

Again, in the equation  $y-y_1 = -\frac{y_1}{2a}(x-x_1)$  of the normal at  $P$ , putting  $y=0$ , we get for the point  $G$ ,  $x-x_1=2a$ , i.e.,  $AG-AN=2a$ , or  $NG=2a$ =half the latus rectum. Hence, *the subnormal of any point of a parabola is constant and is equal to the semi-latus rectum.*

Further,  $\because AT=AN$  and  $AS=AZ$ , by adding we have  $TS=ZN=PM$  (where  $PM$  is the perpendicular on the directrix)  $=SP$ . Hence,  $\angle SPT=\angle PTS$ =the alternate  $\angle TPM$ . Also,  $\because \angle TPG$  is a right angle,  $\angle SPG=\angle SGP$ . Thus, we get further geometrical properties of the parabola, that :

(i) *the tangent at any point on a parabola bisects the angle between the focal distance of the point, and the perpendicular from the point to the directrix.*

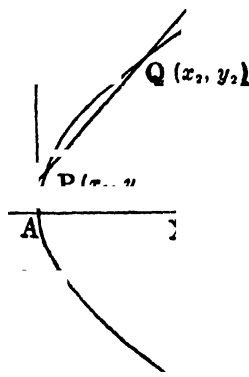
(ii) *the tangent at any point on a parabola makes equal angles with the focal distance of the point and the axis.*

(iii) *the normal at any point on a parabola is equally inclined to the focal distance of the point and the axis.*

**6.7. Length of the chord of the parabola  $y^2=4ax$ , intercepted by the straight line  $y=mx+c$ .**

At the points of intersection of the line with the parabola, both the equations are satisfied. Hence, eliminating  $y$  between

the two equations, the abscissæ of the points of intersection



$$(mx + c)^2 = 4ax,$$

$$\text{or, } m^2x^2 + 2(mc - 2a)x + c^2 = 0, \quad \dots (i)$$

which being a quadratic equation in  $x$ , there are only two values of  $x$  and accordingly only two points of intersection of the straight line with the parabola (which may be real and distinct, real and coincident, or imaginary).

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the two points  $P$  and  $Q$  of intersection. Then  $x_1$  and  $x_2$  are roots of (i).

$$x_1 + x_2 = \frac{2(mc - 2a)}{m^2}, \quad \text{and} \quad x_1 x_2 = \frac{c^2}{m^2}$$

$$\begin{aligned} \therefore (x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= \frac{4(mc - 2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16(a^2 - mca)}{m^4}. \end{aligned}$$

Again,  $P$  and  $Q$  being on the given line,

$$y_1 = mx_1 + c, \quad y_2 = mx_2 + c.$$

$$\therefore y_1 - y_2 = m(x_1 - x_2).$$

length of the chord  $PQ$

$$\begin{aligned} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + m^2)} \\ &= \sqrt{\frac{16(a^2 - mca)}{m^4}(1 + m^2)} = \frac{4}{m^2} \sqrt{a(a - mc)(1 + m^2)} \end{aligned}$$



**Cor. Condition of tangency.**

The given line will touch the parabola only when the two points of intersection come into coincidence, i.e., when the length of the chord intercepted is zero. Hence, the condition that the given line  $y = mx + c$  may touch the parabola  $y^2 = 4ax$  is

$$a - mc = 0, \text{ or, } c = \frac{a}{m}.$$

**6'8.** To show that  $y = mx + \frac{a}{m}$  is a tangent to the parabola  $y^2 = 4ax$  for all values of  $m$ , and to find the point of contact.

The tangent at  $(x_1, y_1)$  of the parabola  $y^2 = 4ax$  is  $yy_1 = 2a(x + x_1)$ , or,  $y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$  (i). If the line  $y = mx + \frac{a}{m}$  (ii) be a tangent to the parabola at  $(x_1, y_1)$ , the equation (i) and (ii) must be the same. Hence, comparing coefficients,

$$\frac{2a}{y_1} = m, \quad \frac{2ax_1}{y_1} = \frac{a}{m}. \quad \therefore x_1 = \frac{a}{m^2}, \quad y_1 = \frac{2a}{m}.$$

The line (ii) therefore will touch the parabola only if the assumed point  $(x_1, y_1)$  is really a point on the parabola  $y^2 = 4ax$ , i.e., if  $\left(\frac{2a}{m}\right)^2 = 4a \cdot \frac{a}{m^2}$  which is evidently satisfied.

Thus,  $y = mx + \frac{a}{m}$  touches the parabola, whatever  $m$  may be, and the point of contact is given by

$$x_1 = \frac{a}{m^2}, \quad y_1 = \frac{2a}{m}.$$

**6'9.** Co-ordinates of a point on the parabola  $y^2 = 4ax$  expressed in terms of a single variable  $t$ .

We notice that if we substitute  $x = at^2$ ,  $y = 2at$  in the equation  $y^2 = 4ax$  of the parabola, the equation is automatically satisfied for all values of  $t$ . Hence, any point on the parabola can have its two co-ordinates expressed in terms of a single variable  $t$  in the form

$$x = at^2, \quad y = 2at.$$

For different points,  $t$  will be different, and for a definite point on the parabola,  $t$  will be definite and unique. A point on the parabola will thus be referred to as the point  $t$ .

In working out many examples in parabola, when given by its standard form  $y^2 = 4ax$ , this assumption of the co-ordinates of a point on it in terms of the single variable  $t$  in the above form will be very helpful.

In this connection we may note that the equation to the tangent to the parabola  $y^2 = 4ax$  at the point  $t$  is (See § 6'4)

$y \cdot 2at = 2a(x + at^2)$ , or,  $y = \frac{x}{t} + at$ . Also the normal at the point  $t$  is (See § 6'5,

$$y - 2at = -\frac{2at}{2a} (x - at^2), \text{ or, } y + tx = 2at + at^3$$

**Note.** Significance of  $t$ .

From the equation of the tangent at  $t$  it is apparent that  $\frac{1}{t}$  is the gradient of the tangent line at  $t$ , i.e.,  $t$  represents the cotangent of the angle made by the tangent line at  $t$  with the  $x$ -axis.

### 6'10. Locus of the middle points of a system of parallel chords ; diameter.

Let  $PQ$ , given by the equation

$$y = mx + c \quad \dots (i)$$

be any one of a system of a parallel chords of the parabola

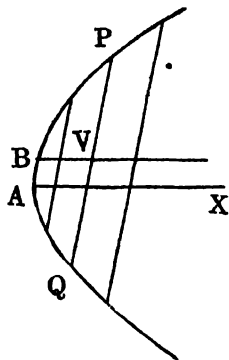
$$y^2 = 4ax \quad \dots (ii).$$

As the chords are parallel,  $m$  is the same for all chords, but  $c$  is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating  $x$ , the ordinates are given by the roots of the equation

$$y^2 = 4a \left( \frac{y - c}{m} \right),$$

$$\text{or, } my^2 - 4ay + 4ac = 0.$$



If  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the two points of intersection  $P$  and  $Q$ , we get  $y_1 + y_2 = \frac{4a}{m}$ . Hence, for the middle point  $V$ ,

$$y = \frac{1}{2}(y_1 + y_2) = \frac{2a}{m}.$$

This being a relation free from  $c$ , it is satisfied by the middle point of every chord of the system.

Hence, this represents the locus of the middle points of the system of parallel chords, and we know that this represents a straight line parallel to the  $x$ -axis.

We thus see that *the locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to its axis.*

Such a straight line is defined to be a *diameter* of the parabola, bisecting the particular system of parallel chords.

For different  $m$ , i.e., for differently directed systems of parallel chords we get different diameters.

**Note.** If  $B$  be the point where the diameter in equation meets the parabola, we have for  $B$  also,  $y = \frac{2a}{m}$ . Hence,  $x = \frac{y^2}{4a} = \frac{a}{m^2}$ . At this point then the tangent line is  $y = mx + \frac{a}{m}$  [See § 6.8], and this is also parallel to the system of chords bisected by the particular diameter.  $B$  is called the vertex of that diameter.

In fact any line parallel to the axis of the parabola is a diameter bisecting all chords parallel to the tangent at its extremity, i.e., at the vertex of that diameter.

### 6.11. Illustrative Examples.

**Ex. 1.** The focus of a parabola is  $(6, 2)$  and its vertex is  $(3, -2)$ . Find the equation to the parabola, and the length of its latus rectum. Also obtain the co-ordinates of the extremities of its latus rectum.

Let  $S$  be the focus having co-ordinates  $(6, 2)$ , and  $A$  the vertex having co-ordinates  $(3, -2)$ . The distance  $AS = \sqrt{(6-3)^2 + (2+2)^2} = 5$ .

Hence, the length of the latus rectum of the parabola =  $4AS = 20$ .

Again, 'm' of the line joining A and S is  $\frac{2-(-2)}{6-8} = \frac{4}{-2}$ , and this line AS is the axis of the parabola. The latus rectum is perpendicular to AS through S, and so its equation is

$$y - 2 = -\frac{1}{2}(x - 6) \quad \dots (i)$$

If  $(x_1, y_1)$  be the co-ordinates of an extremity (say L or L') of the latus rectum,

$\therefore SL = \text{semi-latus rectum} = 10$ ,

$$(x_1 - 6)^2 + (y_1 - 2)^2 = 100 \quad (ii)$$

and as  $(x_1, y_1)$  lies on (i),

$$y_1 - 2 = -\frac{1}{2}(x_1 - 6) \quad (iii)$$

From (ii) and (iii),

$$(x_1 - 6)^2(1 + \frac{1}{4}) = 100,$$

$$\therefore (x_1 - 6)^2 = 64. \therefore x_1 - 6 = \pm 8.$$

Taking + sign,  $x_1 = 14$ , and from (iii),  $y_1 = -4$ .

Taking - sign,  $x_1 = -2$ , and from (iii),  $y_1 = 8$ .

Thus, the co-ordinates of the extremities of the latus rectum are  $(14, -4)$  and  $(-2, 8)$ .

Lastly, produce SA to Z, such that  $AZ = AS$ . Then the co-ordinates of Z being  $(\alpha, \beta)$  say, as A is the mid-point of ZS,

$$\frac{1}{2}(\alpha + 6) = 3 \text{ and } \frac{1}{2}(\beta + 2) = -2.$$

$$\therefore \alpha = 0, \beta = -6.$$

Now since the vertex is the mid-point of the perpendicular from the focus on the directrix, the point Z is clearly the foot of the perpendicular. The directrix of the parabola is therefore the line through Z perpendicular to ZAS, and hence its equation is

$$y + 6 = -\frac{1}{2}(x - 0), \text{ or, } 3x + 4y + 24 = 0. \quad \dots (iv)$$

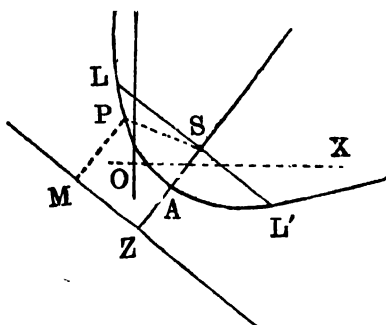
If  $(x, y)$  be the co-ordinates of any point P on the parabola,

$\therefore PS = \text{perp. distance from P on (iv)}$

$$\sqrt{(x-6)^2 + (y-2)^2} = \frac{3x + 4y + 24}{\sqrt{3^2 + 4^2}} = \frac{3x + 4y + 24}{5},$$

whence  $25\{(x-6)^2 + (y-2)^2\} = (3x + 4y + 24)^2$

which is the equation to the parabola.



**Ex. 2.** By suitably transferring the origin, show that the equation  $3y^2 - 10x - 12y - 18 = 0$  reduces to the standard form of equation to a parabola, and hence obtain the co-ordinates of its vertex and focus, and the length of its latus rectum. Also determine the equation to its directrix.

The given equation can be written as

$$3(y^2 - 4y) = 10x + 18, \quad \text{or,} \quad 3(y-2)^2 = 10(x+3)$$

Hence, transferring the origin to  $(-3, 2)$ , the equation reduces to the form

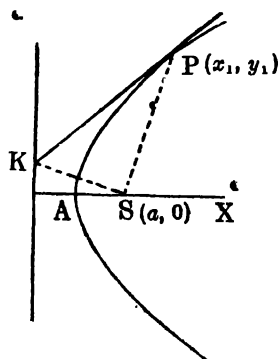
$$y^2 = \frac{10}{3}x \quad \dots \quad \dots \quad \dots \quad (i)$$

which is the standard form of the equation to a parabola. Comparing this with the equation  $y^2 = 4ax$ , whose latus rectum is  $4a$ , vertex is the origin, focus is  $(a, 0)$  and directrix is  $x = -a$ , we see that for the parabola (i), the latus rectum is  $\frac{10}{3}$ , the vertex is the new origin, and referred to this, the co-ordinates of the focus are  $(\frac{5}{3}, 0)$ , and the equation to the directrix is  $x = -\frac{5}{3}$ .

Hence, returning back to the given old origin, the co-ordinates of the vertex are  $(-3, 2)$ , the co-ordinates of the focus are  $(-3 + \frac{5}{3}, 2 + 0)$  i.e.,  $(-2\frac{1}{3}, 2)$ , and the equation to the directrix is  $x = -\frac{5}{3} - 3$  i.e.,  $x = -3\frac{2}{3}$ .

The latus rectum has already been shown to be  $\frac{10}{3}$ .

**Ex. 3.** Prove that the length of any tangent to a parabola intercepted between its point of contact and the directrix subtends a right angle at the focus.



Taking the vertex as origin and axis as  $x$ -axis, let the equation to the parabola be  $y^2 = 4ax$ .  $\dots \dots \dots (i)$

Then its focus  $S$  has co-ordinates  $(a, 0)$  and the equation to the directrix is  $x = -a$ . (ii)

The tangent at any point  $P(x_1, y_1)$  is given by

$$yy_1 = 2a(x + x_1) \quad \dots \quad (iii)$$

This meets the directrix (ii) at  $K$ , whose  $x = -a$

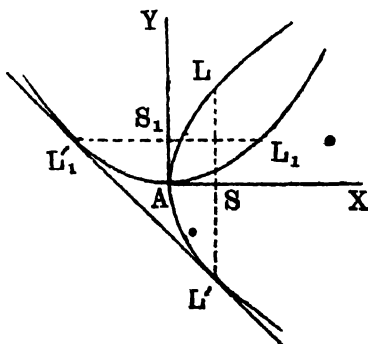
$$\text{and } y = \frac{2a}{y_1}(-a + x_1) \text{ [from (iii)]}.$$

Now the slope ' $m$ ' of the line  $SP = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a}$  and the slope ' $m'$ ' of the

$$\text{line } SK \text{ is } \frac{\frac{2a}{y_1}(x_1 - a) - 0}{-a - a} = \frac{x_1 - a}{y_1}.$$

$\therefore mm' = -1$ . Hence,  $SP$  and  $SK$  are at right angles, i.e.,  $PK$  subtends a right angle at  $S$ .

**Ex. 4.** Two equal parabolas have the same vertex, and their axes are at right angles; prove that their common tangent touches each at an end of its latus rectum.



Let  $y^2 = 4ax$  ... (i) be the equation to one of the parabolas. The other parabola, which is equal to it (and hence has an equal latus rectum  $4a$ ), having the same vertex  $A$  (chosen as origin) and having its axis perpendicular to that of (i) (i.e., along the  $y$ -axis) is then given by

$$x^2 = 4ay \quad \dots \quad (ii)$$

$$\text{Any tangent to (i) is } y = mx + \frac{a}{m} \quad (iii)$$

having the point of contact at  $\frac{a}{m^2}, 2am$ .

If it be a tangent to (ii) also, the two points of intersection of (ii) and (iii) must coincide, or eliminating  $y$ , the two roots of

$$x^2 - 4a \left( nx + \frac{a}{m} \right) = 0 \quad \dots \quad \dots \quad \dots \quad (iv)$$

are equal, which requires

$$(4am)^2 - 4 \cdot \frac{4a^2}{m} = 0,$$

$$\text{or, } m^2 = -1. \quad \therefore m = -1.$$

$\therefore$  the common tangent of (i) and (ii) is  $y = -x - a$ ,

$$\text{or, } x + y + a = 0.$$

The point of contact of this common tangent on (i) is [putting  $m = -1$  here]  $(a, -2a)$ , which is clearly the co-ordinates of one extremity  $L'$  of the latus rectum.

The point of contact of the common tangent on (ii) [  $\therefore$  for the equal roots for (iv) the sum of the roots is  $4am = -4a$  ] is given by  $x = -2a$ , and hence from (ii)  $y = a$ . But  $(-2a, a)$  are clearly the co-ordinates of the extremity  $L'_1$  of the latus rectum of (ii).

Hence, the common tangent of the two parabolas touch each at an end of its latus rectum.

### Examples VI

1. Find the point on the parabola  $y^2 = 18x$  at which the ordinate is three times the abscissa.

2. The parabola  $y^2 = 4ax$  passes through the point  $(2, -6)$ . Find the length of its latus rectum.

3. Find the equation to the line joining the vertex to the positive end of the latus rectum of the parabola  $y^2 = 8x$ .

4. A double ordinate of the parabola  $y^2 = 4ax$  is of length  $8a$ . Prove that the line joining the vertex to its two ends are at right angles. [ H. S. 1960 ]

5. Find the latus rectum of the parabola whose focus is  $(2, -3)$ , and directrix is  $5x - 12y + 6 = 0$ .

6. Find the equation to the parabola

(i) whose focus is  $(5, 3)$  and directrix is  $3x - 4y + 1 = 0$ .

(ii) whose focus is  $(-6, -6)$  and vertex is  $(-2, 2)$ .

## PARABOLA

[ Consider the figure of Ex. 1 of § 6'11.

Equation of the line joining  $S(-6, -6)$  and  $A(-2, 2)$  is

$$\frac{x+6}{-2+6} = \frac{y+6}{2+6} \quad \text{or,} \quad y+6 = 2(x+6).$$

Let the co-ordinates of  $z$  be  $(\alpha, \beta)$ .

$$\therefore \frac{\alpha-6}{2} = -z \quad \text{and} \quad \frac{\beta-2}{2} = z \quad \left[ \text{Since co-ordinates of focus } (-6, -6) \text{ and vertex } A(-2, 2) \right]$$

$$\therefore \alpha = 2, \beta = 10.$$

Equation of directrix which passes through  $z(2, 10)$  and perpendicular to  $AS$  is line (1) is  $y-10 = -\frac{1}{2}(x-2)$ , or,  $x+2y-22=0$ .

Let  $P(x, y)$  any point on the parabola.

$$\therefore SP^2 = (x+6)^2 + (y+6)^2 \quad \text{and} \quad PM = \frac{x+2y-22}{\sqrt{1^2+2^2}}.$$

Since,  $SP^2 = PM^2$ .  $\therefore (x+6)^2 + (y+6)^2 = \frac{(x+2y-22)^2}{5}$  is the equation of parabola which on simplification becomes

$$4x^2 + y^2 - 4xy + 104x + 148y - 12y = 0. ]$$

7. Find the vertex, focus and latus rectum of each of the parabolas

$$(i) \quad y^2 = 4(x+y).$$

$$[ y^2 = 4(x+y) \text{ or } y^2 - 4x = 4x \text{ or } (y-2)^2 = 4(x+1). ]$$

Replacing  $y-2=Y$  and  $x+1=X$ ,  $Y^2=4X$  which is the equation of the parabola with vertex at  $X=0$  and  $Y=0$  i.e.,  $x+1=0$  and  $y-2=0$  i.e.,  $x=-1$ ,  $y=2$  and focus at  $X=1$ ,  $Y=0$  i.e.,  $x+1=1$  and  $y-2=0$ , i.e.,  $x=0$ ,  $y=2$  and latus-rectus=4. ]

$$(ii) \quad x^2 + 2y = 8x - 7.$$

$$x^2 + 2y = 8x - 7 \text{ or } (x-4)^2 = -2(y-\frac{9}{2}) = 2(\frac{9}{2}-y).$$

Replacing  $x-4=X$ ,  $\frac{9}{2}-y=Y$ , equation becomes  $X^2=2Y$  which is the equation of parabola with vertex at  $X=0$ ,  $Y=0$  i.e.,  $x-4=0$  and  $\frac{9}{2}-Y=0$  i.e.,  $x=4$ ;  $y=\frac{9}{2}$ ; focus at  $X=X-4=0$  and  $Y=\frac{9}{2}-y=\frac{1}{2}$ .  $\therefore x=4$ ,  $y=\frac{9}{2}$ . and latus-rectum=2. ]

8. Find out the equation of the tangent to the parabola  $y^2 = 4ax$  at the extremity of the latus rectum. [ H. S. 1960 ]



9. Find the equation to the tangent to the parabola

(i)  $y^2 = 9x$  at the point whose ordinate is 6.

(ii)  $y^2 = 12x$  at the positive extremity of the latus rectum.

[ Equation of latus rectum is  $x = 3$ , for the two extremities of latus-rectum, solve  $y^2 = 12x$  and  $x = 3$ . Co-ordinates of extremities are  $(3, \pm 6)$ . Equation of tangent at positive extremity  $(3, 6)$  is  $y \cdot 6 = 6(x+3)$  or  $y = x + 3$ . ]

10. Show that the foot of the perpendicular from the focus of the parabola  $y^2 = 4ax$  on any tangent lies on the  $y$ -axis.

[ H. S. 1961, Compartmental ]

11. Prove that the tangents at the extremities of the latus rectum of a parabola meet on the directrix, and are at right angles.

12. The two tangents drawn from a point  $P$  to the parabola  $y^2 = 4x$  are at right angles. Find the locus of  $P$ .

13. (i) Prove that any two perpendicular tangents to the parabola  $y^2 = 4ax$  intersect on the directrix.

(ii) If two tangents to a parabola are at right angles, show that their points of contact are at the extremities of a focal chord.

14. A tangent to the parabola  $y^2 = 12x$  makes an angle  $45^\circ$  with the axis. Find the co-ordinates of its point of contact.

15. A tangent to the parabola  $y^2 = 4ax$  makes an angle  $60^\circ$  with the axis. Find its point of contact.

16. Find the equation to the tangent to the parabola  $y^2 = 7x$  which is parallel to the straight line  $x - 4y - 3 = 0$ . Find also its point of contact.

17. Find the equation of the tangent to the parabola  $y^2 = 8x$  which is perpendicular to  $x + 2y + 7 = 0$ .

18. Find the point on the parabola  $y^2 = 8x$  at which the normal is inclined at an angle  $60^\circ$  with the positive direction of the  $x$ -axis.

[ Equation of normal at  $(x_1, y_1)$  to parabola  $y^2 = 8x$  is

$$y - y_1 = -\frac{y_1}{4}(x - x_1).$$

Since this line is inclined at an angle  $60^\circ$  with  $x$ -axis.

$$\therefore -\frac{y_1}{4} = \tan 60^\circ = \sqrt{3}. \quad \therefore y_1 = -4\sqrt{3}.$$

Again since  $(x_1, y_1)$  lies on parabola  $y_1^2 = 8x_1$  i.e.  $(-4\sqrt{3})^2 = 8x_1$ .

$$\therefore x_1 = 6. \quad \text{Hence point of contact of normal is } (6, -4\sqrt{3}).$$

19. Find the equation to the locus of the foot of the perpendicular from the vertex on the tangent at any point of the parabola  $y^2 = 4ax$ .

[ Equation of tangent at  $(x_1, y_1)$  the parabola  $y^2 = 4ax$  is  $yy_1 = 2a(x + x_1)$ .

Equation of normal to above line passing through  $(0, 0)$  is  $y = -\frac{y_1}{2a}x$ .

Solving above two equations to foot of perpendicular is obtained and is given by

$$\frac{y}{2ax_1y_1} = \frac{x}{-4a^2x_1} = \frac{1}{4(x+a)}.$$

Locus of the foot of the perpendicular is obtained by elimination.

From above relation,

$$x(x^2 + y^2) = -\frac{a^2x_1^3}{4(x_1+a)^2} = -ay^2 \quad [\text{Substitute } y_1^2 = 4ax_1]$$

or,  $x(x^2 + y^2) + ay^2 = 0$  which is the reqd locus.]

20. Find the equation to the chord of the parabola  $y^2 = 8x$  which is bisected at the point  $(2, -3)$ .

[ Let  $y = mx + c$  be a chord to parabola  $y^2 = 8x$ .

Eliminating  $x$  between two equations,  $my^2 - 8y + 8c = 0$ .

If  $(x_1, y_1)$  and  $(x_2, y_2)$  are points of intersection of chord with parabola

$$y_1 + y_2 = \frac{8}{m}. \quad \text{But } \frac{1}{2}(y_1 + y_2) = -3 = \frac{8}{m}. \quad \therefore m = -\frac{4}{3}.$$

Hence equation of chord is  $y = -\frac{4}{3}x + c$  and passes through  $(2, -3)$ .

$$\therefore -3 = -\frac{4}{3} \cdot 2 + c. \quad \therefore c = -\frac{1}{3}.$$

Hence equation of chord is  $y = -\frac{4}{3}x - \frac{1}{3}$  or,  $3y + 4x + 1 = 0$ .]

21. Prove that the locus of the middle points of all chords of the parabola  $y^2 = 4ax$  which are drawn through the vertex is the parabola  $y^2 = 2ax$ .

22. Find the length of the chord of the parabola  $y^2 = 12x$  which is inclined at an angle of  $45^\circ$  with the axis, and passes through the point  $(1, 3)$ .

**23.** Find the length of the chord of the parabola  $y^2 = 20x$  along the straight line  $x - 2y + 4 = 0$ .

**24.** Find the length of the normal chord of the parabola  $y^2 = 4ax$  through an extremity of the latus rectum.

**25.** Find the middle point of the line  $3y - 4x = 4$  intercepted by the parabola  $y^2 = 8x$ .

[ Let  $(h, k)$  be mid-point.

$$\therefore \frac{4}{3} = 3 \quad [\text{Note } \S 6.10]$$

Also since  $(h, k)$  lies on  $3y - 4x = 4$ ,

$$\therefore 3k - 4h = 4 \quad \text{or, } 4h = 3k - 4 = 9 - 4 = 5, \quad \therefore h = \frac{5}{4}.$$

$\therefore$  Required middle-point  $(\frac{5}{4}, 3)$  ]

**26.** Prove that the product of the ordinates of the extremities of a focal chord of a parabola is constant, and deduce that the normals at the extremities of any focal chord are at right angles.

[ Let the equation of parabola be  $y^2 = 4ax$ . Equation of any chord which passes through focus  $(a, 0)$  i. e. equation of focal chord is  $y = m(x - a)$ .

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of extremities of focal chord.

Solving  $y^2 = 4ax$  and  $y = m(x - a)$ , the co-ordinates of extremities can be obtained, and eliminating  $x$ , between two equations,  $my^2 - 4ay - 4a^2m = 0$ .

Product of ordinates of extremities of focal chord

$$\equiv y_1 y_2 = -\frac{4a^2 m}{m} = -4a^2 \equiv \text{constant.}$$

Again equations of normals at  $(x_1, y_1)$  and  $(x_2, y_2)$  are

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{and} \quad y - y_2 = -\frac{y_2}{2a}(x - x_2),$$

' $m$ ' of the first line being  $m_1 = -\frac{y_1}{2a}$  and that of second being  $m_2 = -\frac{y_2}{2a}$ .

$$\therefore m_1 m_2 = +\frac{y_1 y_2}{4a^2} = -\frac{4a^2}{4a^2} = -1.$$

Hence normals at extremities of focal chord are perpendicular. ]

**27.** Prove that the normal chord of a parabola at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.

[ Let the equation of parabola be  $y^2 = 4ax$  and the co-ordinates of point  $P$  on parabola at which ordinate is equal to abscissa is  $P(4a, 4a)$ .

Equation of normal chord at  $(4a, 4a)$  is  $y - 4a = -\frac{4a}{2a}(x - 4a)$  or  $y + 2x = 12a$ .

Solving  $y^2 = 4ax$  and  $y + 2x = 12a$ , the points at which chords intersect parabola are obtained and co-ordinates of points of intersection are  $P(4a, 4a)$  and  $Q(9a, -6a)$ .

$SP^2 = 25a^2$ ;  $SQ^2 = 100a^2$ ;  $PQ^2 = 125a^2$  where  $S$  is focus  $(a, 0)$

$\therefore SP^2 + SQ^2 = PQ^2$ . Hence the result.

28. Find the equation to the common tangent of the parabolas  $y^2 = 32x$  and  $x^2 = 4y$ .

29. Prove that the common tangents of the parabola  $y^2 = 4ax$  and the circle  $x^2 + y^2 - 2ax = 3a^2$  are both inclined at  $30^\circ$  to the  $x$ -axis.

30. Show that the sum of the ordinates of the extremities of any chord of a parallel system is constant.

[ Consider chords parallel to  $y = mx$ .

Equation of any chord parallel to  $y = mx$  is  $y = mx + c$ .

Solving  $y = mx + c$  and  $y^2 = 4ax$ , co-ordinates of extremities are obtained.

Eliminating  $x$  between two equations,

$$y^2 = 4 \frac{a}{m} (y - c) \quad \text{or,} \quad my^2 - 4ay + \frac{4ac}{m} = 0.$$

If  $(x_1, y_1)$  and  $(x_2, y_2)$  are co-ordinates of extremities

$$y_1 + y_2 = \frac{4a^2}{m^2} = \text{constant}.$$

[ Since  $m$  for all parallel chords remain same. ]

### ANSWERS

1.  $(2, 6)$ .
2. 18.
3.  $y = 2x$ .
5. 8.
6. (i)  $25\{(x-5)^2 + (y-3)^2\} = (3x-4y+1)^2$ .
- (ii)  $4x^2 - 4xy + y^2 + 104x + 148y - 124 = 0$ .
7. (i)  $(-1, 2)$ ;  $(0, 2)$ ; 4.
- (ii)  $(4, \frac{4}{3})$ ;  $(4, 4)$  2.
8.  $y = \pm(x+a)$ .
9. (i)  $3x - 4y + 12 = 0$ .
- (ii)  $y = x + 3$ .
12.  $x = -1$ .
14.  $(3, 6)$ .
15.  $(\frac{a}{8}, \frac{2a}{\sqrt{3}})$ .
16.  $x - 4y + 28 = 0$ ;  $(28, 14)$ .
17.  $y = 2x + 1$ .
18.  $(6, -4\sqrt{3})$ .
19.  $x(x^2 + y^2) + ay^2 = 0$ .
20.  $4x + 3y + 1 = 0$ .
22.  $4\sqrt{6}$ .
23. 80.
24.  $8a\sqrt{2}$ .
25.  $(\frac{1}{2}, 3)$ .
26.  $2x + y + 4 = 0$ .

## CHAPTER VII

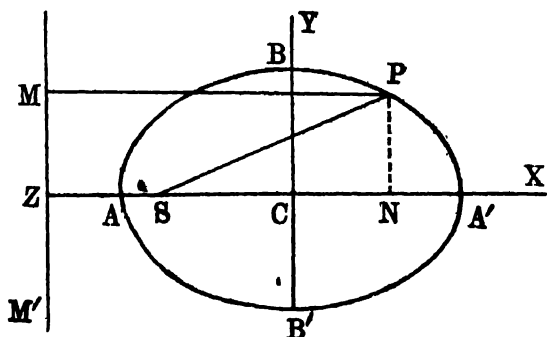
### ELLIPSE

#### 7.1. Ellipse.

An *ellipse* is a curve traced out by a point which moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the ratio being less than unity.

The fixed point is called the *focus*, the fixed straight line is called the *directrix*, and the constant ratio (less than unity in this case) is called the *eccentricity* of the ellipse.

#### 7.2. Standard equation of an ellipse.



Let  $S$  be the focus,  $MM'$  the directrix, and  $e (< 1)$  the given eccentricity of the ellipse.

Draw  $SZ$  perpendicular from  $S$  on  $MM'$ , and let it be divided internally at  $A$  and externally at  $A'$  in the ratio  $e : 1$ . As  $e < 1$ ,  $SA' < A'Z$ , and accordingly  $A'$  is to the right of  $S$  as in the figure, on the same side of the directrix  $MM'$  as  $A$ ,  $S$  being between  $A$  and  $A'$ . Then  $SA = e.AZ$  and  $SA' = e.A'Z$ . Hence, by definition of the ellipse,  $A$  and  $A'$  are points on the ellipse.

Let  $C$  be the middle point of  $AA'$ .

Thus,  $SA + SA' = e(AZ + A'Z)$  and  $SA' - SA = e(A'Z - AZ)$ .

Hence,  $AA'$  or  $2CA = e \cdot 2CZ$  and  $2CS = e \cdot AA' = e \cdot 2CA$ .

Let  $CA(=CA') = a$ .

Then  $CZ = \frac{a}{e}$  and  $CS = ae$ . Let us choose  $C$  as origin, and  $CX$  along  $AA'$  as  $x$ -axis, the  $y$ -axis  $CY$  being  $B'CB$  perpendicular to  $AA'$  through  $C$ .

Now  $P$  being any point on the ellipse whose co-ordinates are  $(x, y)$ , let  $PN$  be the perpendicular from  $P$  to the  $x$ -axis  $AA'$ , and  $PM$  be perpendicular to the directrix  $MM'$ . Then  $CN = x$ ,  $PM = ZN = ZC + CN = \frac{a}{e} + x$ . Also co-ordinates of  $S$  are evidently  $(-ae, 0)$  [ $\because CS = ae$ ].

Hence, from the property of the ellipse,

$$SP = e \cdot PM \quad \text{or} \quad SP^2 = e^2 \cdot PM^2.$$

$$(x + ae)^2 + y^2 = e^2 \left( \frac{a}{e} + x \right)^2$$

$$\text{or, } x^2(1 - e^2) + y^2 = a^2(1 - e^2), \quad (\because e < 1 \text{ here})$$

$$\text{or, writing } a^2(1 - e^2) = b^2,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots \quad \dots \quad \dots \quad (1)$$

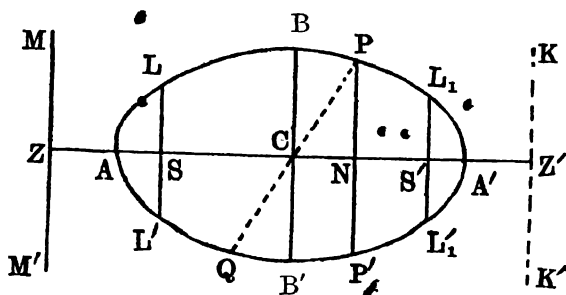
This being the relation satisfied by the co-ordinates of any point on the ellipse, it represents the equation to the ellipse in its standard form.

Here,  $C$ , the middle point of  $AA'$  (called the *centre*) is the origin,  $CA = CA' = \frac{1}{2}AA' = a$ , and  $b^2 = a^2(1 - e^2)$ .

### 7.3. Shape and elementary properties of the ellipse.

From the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , it is apparent that corresponding to any value of  $x$ , there are two equal and opposite values of  $y$ , namely  $\pm \frac{b}{a} \sqrt{a^2 - x^2}$ . Hence, on a line perpendicular to  $AA'$ , corresponding to any point  $P$  on one side of it, there is another symmetrical point  $P'$  on the other side. Thus

every line perpendicular to  $AA'$  is bisected by it, and accordingly the curve is symmetrical with respect to the  $x$ -axis.



Similarly, for every value of  $y$  we get two equal and opposite values of  $x$ , and thus the curve is symmetrical with respect to the  $y$ -axis also. Accordingly if we take points  $S', Z'$  on the  $x$ -axis, such that  $CS' = CS$  and  $CZ' = CZ$  on opposite sides of  $C$ , and draw  $KZ'K'$  parallel to  $MZM'$ , the ellipse, from symmetry about  $BCB'$ , can as well be described with  $S'$  as focus and  $KK'$  as directrix,  $e$  being the same. Thus there is a second focus and a second directrix of the ellipse symmetrically situated with respect to  $C$ .

Again, from the equation to the ellipse, for  $y=0$  we get  $x = \pm a$ . Hence, the ellipse cuts the  $x$ -axis at points  $A'$  and  $A$  given by  $x=a$  and  $x=-a$  respectively. Similarly, for  $x=0$ , we get  $y = \pm b$ , so that the ellipse cuts the  $y$ -axis at  $B$  and  $B'$  given by  $y=b$  at  $B$  and  $y=-b$  at  $B'$ , so that  $CB = CB' = b$  in length.

Moreover, from the equation of the ellipse, if  $x > a$  or  $x < -a$ ,  $\frac{x^2}{a^2} > 1$  and so  $y^2$  is negative, and hence  $y$  is imaginary. Thus there are no points of the ellipse beyond  $A'$  to the right, or beyond  $A$  to the left. Similarly, if  $y > b$  or  $y < -b$ ,  $x$  is imaginary, and thus there are no points of the ellipse above  $B$  or below  $B'$  in the  $y$ -direction. Hence, the ellipse is limited in all directions, and is a closed curve.

Lastly, from the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  of the ellipse, if  $(x_1, y_1)$  be the co-ordinates of a point  $P$  on the ellipse which satisfy the

equation, the co-ordinates  $(-x_1, -y_1)$  will also satisfy it, and accordingly the diagonally opposite point  $Q$ , where  $PQ$  is bisected at  $C$ , is also a point on the ellipse. Thus *every chord of the ellipse through  $C$  is bisected at  $C$* , and thus the ellipse is symmetrical with respect to the origin  $C$ , the mid-point of  $AA'$  or  $BB'$ . That is why  $C$  is called the *centre* of the ellipse.

The length  $AA' = 2a$  along the  $x$ -axis is called the *major axis* of the ellipse.

The length  $BB' = 2b$  along the  $y$ -axis is called the *minor axis* of the ellipse.

The points  $A, A', B, B'$  are called the *vertices* of the ellipse.

The chord  $LSL'$  through the focus  $S$ , or  $L_1S'L_1$  through the focus  $S'$ , perpendicular to the major axis  $AA'$  (i.e., parallel to the directrix) is called the *latus rectum* of the ellipse, both being of same length by symmetry.

Now  $ae$  being the length  $CS'$ , the  $x$ -co-ordinate of the extremity  $L_1$  or  $L'_1$  of the latus rectum is  $ae$ . Hence, from the equation to the ellipse, the  $y$ -co-ordinate of  $L_1$  or  $L'_1$  is given by  $\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$ .

$$\text{Hence, } y = \pm b\sqrt{1-e^2} = \pm a(1-e^2).$$

Thus the length  $L_1L'_1$ , or  $LL'$  of the latus rectum

$$= 2a(1-e^2) = 2\frac{b^2}{a}.$$

$$\therefore \text{Semi-latus rectum} = \frac{b^2}{a} = a(1-e^2).$$

Co-ordinates of the extremity  $L_1$ , of the latus rectum are  $[ae, a(1-e^2)]$ .

The eccentricity of the ellipse is given by

$$b^2 = a^2(1-e^2) \text{ or } e^2 = \frac{a^2-b^2}{a^2}.$$

**Lengths of the focal distances  $S'P, SP$  of any point  $P$  on the ellipse :**



Let the co-ordinates of  $P$  be  $(x_1, y_1)$ . Those of  $S'$  being  $(ae, 0)$ , we get

$$\begin{aligned} S'P^2 &= (x_1 - ae)^2 + y_1^2 = (x_1 - ae)^2 + b^2 \left(1 - \frac{x_1^2}{a^2}\right) \\ &\quad \text{[ from the equation to the ellipse ]} \\ &= (x_1 - ae)^2 + (1 - e^2)(a^2 - x_1^2), \\ &\quad \text{[ } \because b^2 = a^2(1 - e^2) \text{ ]} \\ &= e^2 x_1^2 - 2x_1 ae + a^2 = (a - ex_1)^2. \end{aligned}$$

$\therefore S'P = a - ex_1$ , which is the positive value of  $S'P$ ,  $\because x_1 < a$ , as also  $e < 1$ .

Similarly,  $SP = a + ex_1$ .

Thus,  $SP + S'P = 2a = \text{length of the major axis}$ . Hence, we get an important property of the ellipse, namely, *the sum of the focal distances of any point on the ellipse is constant and equal to the major axis*.

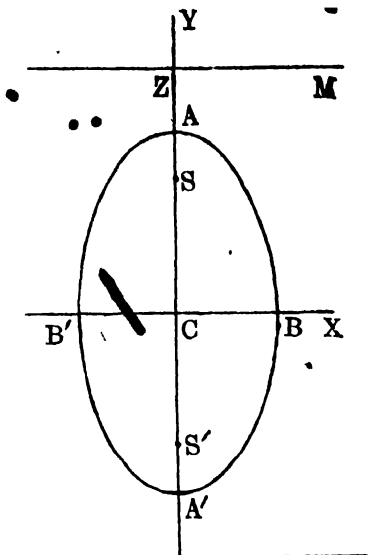
**Cor.** Focal distance of an extremity of the minor axis is equal to the semi-major axis.

**Note 1.** As the whole figure is symmetrical with respect to the minor axis  $PCB'$ , henceforth, for convenience, as a matter of convention, we shall denote the right-hand focus  $(ae, 0)$  as  $S$ , the right-hand vertex  $(a, 0)$  as  $A$ , and the left-hand focus  $(-ae, 0)$  as  $S'$ , the left-hand vertex  $(-a, 0)$  as  $A'$ , the right-hand directrix having equation  $x = \frac{a}{e}$  being denoted by  $MM'$ , and left-hand directrix  $x = -\frac{a}{e}$  as  $KZ'K'$ .

**Note 2.** The equation  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ ,  $a > b$ .

If in the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , the axes of  $x$  and  $y$  be interchanged, the equation becomes  $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$  or  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ . Here,  $a$  being greater than  $b$ , the major axis having length  $2a$  is along the  $y$ -axis, and the minor axis of

length  $2b$  is along the  $y$ -axis. The foci, being on the major axis, i.e.,  $y$ -axis, will have co-ordinates,  $(0, \pm \sqrt{a^2 - b^2})$ . The eccentricity is as before



$e = \sqrt{(a^2 - b^2)}/a$ . The directrices being parallel to the minor axis i.e.,  $x$ -axis here, are given by  $y = \pm \frac{a}{e}$ .

**7.4. Equation to the tangent at a given point  $(x_1, y_1)$  on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .**

Let  $P$  be the point  $(x_1, y_1)$  on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (i)$$

and let  $Q(x_2, y_2)$  be a neighbouring point on it. The equation to the chord  $PQ$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad (ii)$$

Now since  $P$  and  $Q$  both lie on the ellipse (i), we have,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \dots \quad \dots \quad \text{(iii)}$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1. \quad \dots \quad \dots \quad \text{(iv)}$$

Hence, subtracting,

$$\frac{x_2^2 - x_1^2}{a^2} + \frac{y_2^2 - y_1^2}{b^2} = 0, \quad \text{or,} \quad \frac{y_2 - y_1}{x_2 - x_1} = -\frac{b^2}{a^2} \cdot \frac{x_2 + x_1}{y_2 + y_1}.$$

$\therefore$  equation (ii) can be written as

$$y - y_1 = -\frac{b^2}{a^2} \cdot \frac{x_2 + x_1}{y_2 + y_1} (x - x_1) \quad \dots \quad \text{(v)}$$

Now make  $Q$  approach  $P$  and ultimately coincide with it, so that the co-ordinates  $(x_2, y_2)$  coincide with  $(x_1, y_1)$ . In that limiting position, the straight line  $PQ$  becomes the tangent at  $P$ , whose equation [from (v)] then becomes

$$y - y_1 = -\frac{b^2}{a^2} \cdot \frac{x_1}{y_1} (x - x_1),$$

$$\text{or,} \quad (y - y_1) \frac{y_1}{b^2} + \frac{x_1}{a^2} (x - x_1) = 0,$$

$$\text{or,} \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1. \quad [\text{by (iii)}]$$

Hence, the equation to the tangent at  $(x_1, y_1)$  to the ellipse (i) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

7.5. Equation to the normal at  $(x_1, y_1)$  to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

The tangent at  $(x_1, y_1)$  to the ellipse is  $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ,

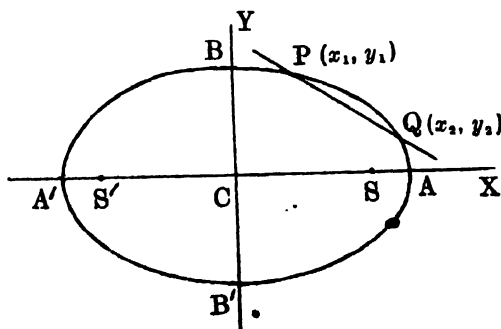
$$\text{or, } y = -\frac{b^2 x_1}{a^2 y_1} x + \frac{b^2}{y_1}, \quad \text{of which the 'm' is } -\frac{b^2 x_1}{a^2 y_1}.$$

The normal, which is perpendicular to the tangent through  $(x_1, y_1)$ , has its 'm' =  $\frac{a^2 y_1}{b^2 x_1}$ , and accordingly its equation is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1),$$

or, 
$$\frac{-x_1}{\frac{y_1}{b^2}} = \frac{y - y_1}{\frac{a^2 y_1}{b^2 x_1}}.$$

7.6. Length of the chord of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , intercepted by the straight line  $y = mx + c$ .



At the points of intersection of the line with the ellipse, both the equations are satisfied. Hence, eliminating  $y$  between the two equations, the abscissæ of the points of intersection will be given by

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

$$\text{or, } x^2 \left( \frac{1}{a^2} + \frac{m^2}{b^2} \right) + \frac{2mc}{b^2} x + \left( \frac{c^2}{b^2} - 1 \right) = 0, \quad (i)$$

which being a quadratic equation in  $x$ , there are only two values of  $x$  and accordingly, only two points of intersection of the given straight line with the ellipse (real, coincident, or imaginary).

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the two points  $P$  and  $Q$  of intersection. Then  $x_1$  and  $x_2$  are the roots of (i).

$$\therefore x_1 + x_2 = -\frac{2mc}{b^2} / \left( \frac{1}{a^2} + \frac{m^2}{b^2} \right) = -\frac{2mca^2}{a^2m^2 + b^2}$$

$$\text{and } x_1x_2 = \left( \frac{c^2}{b^2} - 1 \right) / \left( \frac{1}{a^2} + \frac{m^2}{b^2} \right) = \frac{a^2(c^2 - b^2)}{a^2m^2 + b^2}$$

$$\begin{aligned} \therefore (x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= \frac{4m^2c^2a^4}{(a^2m^2 + b^2)^2} - \frac{4a^2(c^2 - b^2)}{a^2m^2 + b^2} \\ &= \frac{4a^2\{m^2c^2a^2 - (c^2 - b^2)(a^2m^2 + b^2)\}}{(a^2m^2 + b^2)^2} \\ &= \frac{4a^2b^2(a^2m^2 + b^2 - c^2)}{(a^2m^2 + b^2)^2} \end{aligned}$$

Again  $P$  and  $Q$  lying on the given line,

$$y_1 = mx_1 + c, \quad y_2 = mx_2 + c. \quad \therefore y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$  length of the chord  $PQ$

$$\begin{aligned} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + m^2)} \\ &= \sqrt{\frac{4a^2b^2(a^2m^2 + b^2 - c^2)}{(a^2m^2 + b^2)^2}(1 + m^2)} \\ &= \frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2} \end{aligned}$$

**Cor. Condition of tangency.**

The given line will touch the ellipse only when the two points of intersection come into coincidence, i.e., when the length of the chord intercepted is zero. Hence, the condition that the given line  $y = mx + c$  may touch the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is  $a^2m^2 + b^2 - c^2 = 0$ , or,  $c = \pm \sqrt{a^2m^2 + b^2}$ .

**7.7.** To show that  $y = mx + \sqrt{a^2m^2 + b^2}$  is a tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  for all values of  $m$ , and to find the point of contact.

The tangent at  $(x_1, y_1)$  of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1, \text{ or, } y = -\frac{b^2 x_1}{a^2 y_1} x + \frac{b^2}{y_1}. \quad \dots (i)$$

If the line  $y = mx + \sqrt{a^2 m^2 + b^2}$  ... (ii) be a tangent to the ellipse at  $(x_1, y_1)$ , the equations (i) and (ii) must be same. Hence, comparing coefficients,

$$-\frac{b^2 x_1}{a^2 y_1} = m, \quad \frac{b^2}{y_1} = \sqrt{a^2 m^2 + b^2}.$$

$$\therefore y_1 = \frac{b^2}{\sqrt{a^2 m^2 + b^2}}, \quad x_1 = -\frac{a^2 m y_1}{b^2} = -\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}.$$

The line (ii) therefore will touch the ellipse only if the assumed point  $(x_1, y_1)$  is really a point on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,

$$\text{i.e., if } \left(-\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}\right)^2 + \left(\frac{b^2}{\sqrt{a^2 m^2 + b^2}}\right)^2 = 1$$

which is evidently satisfied.

Thus,  $y = mx + \sqrt{a^2 m^2 + b^2}$  touches the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , whatever  $m$  may be, and the point of contact is given by

$$x_1 = -\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \quad y_1 = \frac{b^2}{\sqrt{a^2 m^2 + b^2}}.$$

Similarly,  $y = mx - \sqrt{a^2 m^2 + b^2}$  is also a tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , for all values of  $m$ , the co-ordinates of the point of contact being

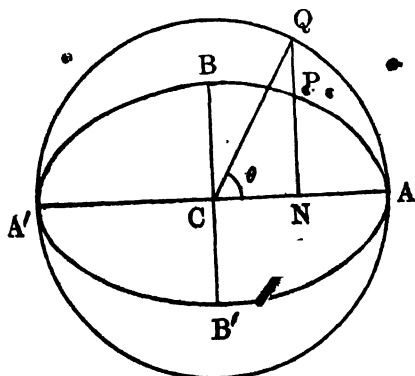
$$\frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \quad -\frac{b^2}{\sqrt{a^2 m^2 + b^2}}.$$

### 7'8. Auxiliary circle.

The circle on the major axis  $AA'$  of an ellipse as diameter is defined as the *auxiliary circle* of the ellipse.

The centre of the circle being the origin  $C$ , and its radius being the semi-major axis  $a$ , the equation to the auxiliary circle is

$$x^2 + y^2 = a^2.$$



Let  $PN$  be an ordinate of the ellipse, which, when produced, meets the auxiliary circle at  $Q$ .

Then  $x$  being the abscissa  $CN$  of the point  $P$ , from the equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  of the ellipse, the ordinate of the ellipse

$$PN = y = \sqrt{b^2 \left(1 - \frac{x^2}{a^2}\right)} = \frac{b}{a} \sqrt{a^2 - x^2}$$

Also, from the equation  $x^2 + y^2 = a^2$  of the auxiliary circle, the abscissa  $CN = x$  being the same, the ordinate  $CN = \sqrt{a^2 - x^2}$

$$\text{Thus, } \frac{PN}{QN} = \frac{b}{a}.$$

Hence, the ratio of any ordinate of the ellipse to the corresponding ordinate of its auxiliary circle is always the same, and is equal to the ratio of the minor axis to the major axis of the ellipse.

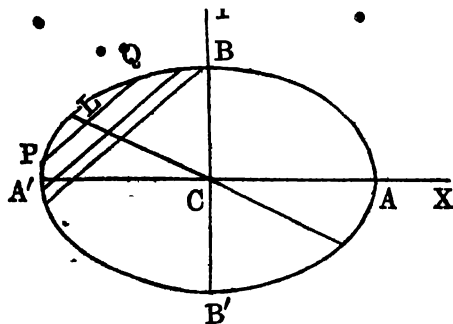
**Note.** Co-ordinates of any point on an ellipse expressed in terms of a single variable; *eccentric angle* of a point on the ellipse.

Let  $\angle QCN = \theta$ . Then,  $\because CQ = a$ , clearly  $CN = a \cos \theta$ ,  $NQ = a \sin \theta$ .  
 $\therefore NP = \frac{b}{a} \cdot a \sin \theta = b \sin \theta$ . Thus, co-ordinates of any point  $P$  on the

## ELLIPSE

ellipse can be written as  $a \cos \theta$ ,  $b \sin \theta$ , in terms of a single variable. Here,  $\theta$  is called the *eccentric angle* of the point  $P$  on the ellipse.

**7.9. Locus of the middle points of a system of parallel chords ; diameter.**



Let  $PQ$ , given by the equation  $y = mx + c$  ... (i)

be any one of a system of parallel chords of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \dots \dots \dots \text{(ii)}$$

As the chords are parallel,  $m$  is the same for all chords, but  $c$  is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating  $y$ , the abscissæ are given by the roots of the equation

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

$$\text{or, } (a^2 m^2 + b^2)x^2 + 2a^2 mcx + a^2(c^2 - b^2) = 0 \quad \dots \text{(iii)}$$

Thus, if  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of  $P$  and  $Q$ ,  $x_1, x_2$  are the roots of (iii), and so  $x_1 + x_2 = -\frac{2a^2 mc}{a^2 m^2 + b^2}$ . Hence, if  $(X, Y)$  be the co-ordinates of the mid-point  $L$  of  $PQ$ ,

$$X = \frac{1}{2}(x_1 + x_2) = -\frac{a^2 mc}{a^2 m^2 + b^2}.$$

Also,  $\because L$  is a point on (i),  $Y = mX + c$ .



$\therefore$  eliminating  $c$ ,

$$Y = mX - \frac{a^2 m^2 + b^2}{a^2 m} X = -\frac{b^2}{a^2 m} X,$$

which is independent of  $c$ , and so holds for the middle point of any chord of the parallel system.

Hence, the locus of middle points of a system of parallel chords of the ellipse, parallel to  $y = mx$ , is

$$y = -\frac{b^2}{a^2 m} x$$

which is evidently a straight line passing through the origin, i.e., the centre  $C$  of the ellipse. This straight line is called a *diameter* of the ellipse. For different values of  $m$  (i.e., for differently directed system of parallel chords) we get different diameters, all passing through the centre.

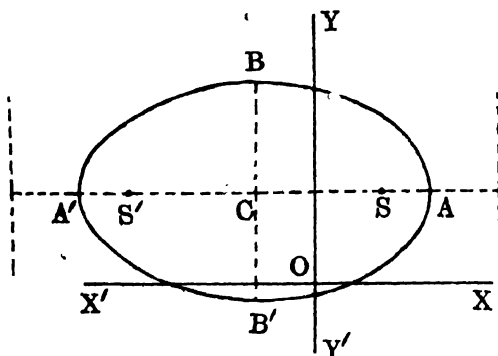
### 7'10. Illustrative Examples.

**Ex. 1.** Show that the equation  $5x^2 + 9y^2 + 10x - 36y - 4 = 0$  represents an ellipse, and find its eccentricity, latus rectum, and co-ordinates of the foci. Find also the equations to its directrices.

The given equation can be written as

$$5(x^2 + 2x) + 9(y^2 - 4y) = 4, \quad \text{or,} \quad 5(x+1)^2 + 9(y-2)^2 = 45,$$

$$\text{i.e.,} \quad \frac{(x+1)^2}{9} + \frac{(y-2)^2}{5} = 1.$$



Transferring the origin to the point  $(-1, 2)$ , the equation reduces to

$$\frac{x^2}{9} + \frac{y^2}{5} = 1 \quad \dots \quad \dots \quad \dots \quad (i)$$

which is the standard form of the equation to an ellipse, with centre as origin.

Hence, the given equation represents an ellipse whose centre (the new origin) has co-ordinates  $(-1, 2)$  referred to the original axes of co-ordinates.

Comparing equation (i) with the standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ we notice that for (i), } a^2 = 9, b^2 = 5.$$

Thus, eccentricity of the given ellipse is

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{9-5}{9}} = \frac{2}{3}.$$

$$\text{Latus rectum} = 2 \frac{b^2}{a} = 2 \cdot \frac{5}{3} = 3\frac{1}{3}.$$

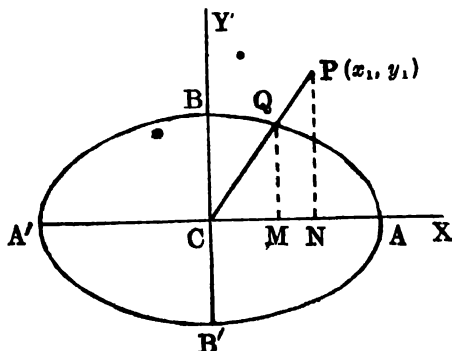
Co ordinates of the foci referred to the centre are

$$(\pm ae, 0), \text{ i.e., } (\pm \frac{2}{3} \cdot 3, 0), \text{ or } (\pm 2, 0).$$

Hence, referred to the original axes, the co-ordinates of the foci are  $(-1+2, 2+0)$  and  $(-1-2, 2+0)$  i.e.,  $(1, 2)$  and  $(-3, 2)$  respectively.

Also, the equations to the directrices referred to centre are  $x = \pm \frac{a}{e}$  or,  $x = \pm \frac{3}{\frac{2}{3}} = \pm \frac{9}{2}$ . Hence, referred to original axes, the equations to the directrices are  $x = \pm \frac{9}{2} - 1$  i.e.,  $x = \frac{7}{2}$  and  $x = -\frac{11}{2}$  respectively.

**Ex. 2.** Prove that the point  $(x_1, y_1)$  is inside or outside the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  according as  $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$  or  $> 1$ .



Let  $P$  be the point where co-ordinates are  $(x_1, y_1)$  and let  $CP$ , the line joining the centre (origin) to  $P$  intersect the ellipse at  $Q$ . Then, if  $\frac{CP}{CQ} = \lambda$ , clearly  $P$  is outside the ellipse if  $\lambda > 1$  and inside if  $\lambda < 1$ .

Now,  $PN$  and  $QM$  being perpendiculars on the  $x$ -axis  $CAX$ , clearly  $x_1 = CN$ ,  $y_1 = NP$ , and  $\frac{CM}{CN} = \frac{MQ}{NP} = \frac{CQ}{CP} = \frac{1}{\lambda}$ .

$\therefore$  the co-ordinates of  $Q$ , namely  $CM$  and  $MQ$  are respectively  $\frac{x_1}{\lambda}$ ,  $\frac{y_1}{\lambda}$ .

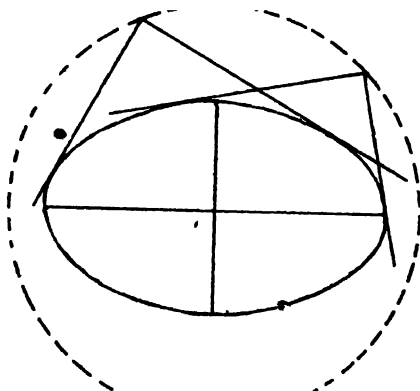
As  $Q$  lies on the ellipse, its co-ordinates will satisfy the equation to the ellipse, and hence

$$\frac{x_1^2}{\lambda^2 a^2} + \frac{y_1^2}{\lambda^2 b^2} = 1, \text{ or, } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \lambda^2.$$

Hence, the point  $P$  is outside or inside the ellipse according as  $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$  or  $< 1$ .

**Ex..3.** Prove that the locus of point of intersection of any two perpendicular tangents to an ellipse is a circle.

Let  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots (i)$  be the equation to the ellipse.



Any tangent to (i) is  $y = mx + \sqrt{a^2 m^2 + b^2} \dots (ii)$ . For the perpendicular tangent, replacing  $m$  by  $-\frac{1}{m}$ , the equation is  $y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} + b^2}$ ,  
 $my = -x + \sqrt{a^2 + b^2 m^2} \dots (iii)$

At the point of intersection of (ii) and (iii) both the equations are satisfied. Hence, if we eliminate  $m$  between these two equations, the re-

tion obtained will be satisfied at the points of intersection of every such pair of perpendicular tangents, and will thus represent the equation to the desired locus.

Now, from (ii) and (iii),

$$y - mx = \sqrt{a^2 m^2 + b^2} \text{ and } my + x = \sqrt{a^2 + b^2 m^2}.$$

Squaring and adding,

$$(x^2 + y^2)(1 + m^2) = (a^2 + b^2)(1 + m^2).$$

$$\therefore x^2 + y^2 = a^2 + b^2$$

which evidently represents a circle with its centre at the centre of the ellipse.

**Note.** The circle is known as the **director circle** of the ellipse.

**Ex. 4.** Find the length of the chord of the ellipse  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ , whose middle point is  $(\frac{1}{2}, \frac{3}{2})$ .

Let the equation to the chord  $PQ$ , whose middle point is  $(\frac{1}{2}, \frac{3}{2})$  be

$$y - \frac{3}{2} = m(x - \frac{1}{2}), \text{ or } y = mx + \frac{4 - 5m}{10}. \quad \dots \quad \dots \quad (i)$$

The abscissae of its points of intersection  $P$  and  $Q$  with the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \quad \dots \quad \dots \quad \dots \quad (ii)$$

are given by [ eliminating  $y$  between (i) and (ii) ]

$$\frac{x^2}{25} + \frac{1}{16} \left\{ mx + \frac{4 - 5m}{10} \right\}^2 = 1,$$

$$\text{or, } (16 + 25m^2)x^2 + 5m(4 - 5m)x + \frac{(4 - 5m)^2 - 1600}{4} = 0. \quad \dots \quad (iii)$$

Now, if  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of  $P$  and  $Q$ , then  $x_1, x_2$  are the roots of (iii).

$$\text{Hence, } x_1 + x_2 = \frac{5m(5m - 4)}{16 + 25m^2} \quad \dots \quad \dots \quad (iv)$$

$$\text{and } x_1 x_2 = \frac{(4 - 5m)^2 - 1600}{4(16 + 25m^2)}. \quad \dots \quad \dots \quad (v)$$

But the abscissa of the middle point of  $PQ$  is given to be  $\frac{1}{2}$ .

$$\therefore \frac{1}{2}(x_1 + x_2) = \frac{1}{2}, \text{ or, } x_1 + x_2 = 1.$$

$$\therefore \text{ from (iv), } 16 + 25m^2 = 5m(5m - 4). \quad \therefore m = -\frac{4}{5}.$$

$$\therefore (v) \text{ gives } x_1 x_2 = \frac{64 - 1600}{4 \cdot 92} = -12.$$

$$\therefore (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2 = 1 + 48 = 49. \quad \dots \quad \dots \quad (vi)$$

As  $P$  and  $Q$  both lie on (i),

$$y_1 - \frac{2}{3} = m(x_1 - \frac{1}{3}) \quad y_2 - \frac{2}{3} = m(x_2 - \frac{1}{3}).$$

$$\therefore y_1 - y_2 = m(x_1 - x_2) = -\frac{2}{3}(x_1 - x_2).$$

$$\begin{aligned} \therefore PQ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2 (1 + \frac{4}{9})} \\ &= \sqrt{49 \times \frac{13}{9}} = \frac{7}{3} \sqrt{41}. \end{aligned}$$

**Ex. 5.** Prove that in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , if the line  $y = m'x$  bisects all chords parallel to  $y = mx$ , then  $y = mx$  also bisects all chords parallel to  $y = m'x$ .

As in § 7.9, we see that the bisector of all chords parallel to  $y = mx$  is the diameter  $y = -\frac{b^2}{a^2 m} x$ . Hence, if this diameter is given to be  $y = m'x$ , we must have  $m' = -\frac{b^2}{a^2 m}$  or  $mm' = -\frac{b^2}{a^2}$  ... (i), which is the condition that  $y = m'x$  may bisect all chords parallel to  $y = mx$ .

Similarly, the condition that  $y = mx$  may bisect all chords of the ellipse parallel to  $y = m'x$  is  $m'm = -\frac{b^2}{a^2}$ , which is identical with (i).

Hence, if  $y = m'x$  bisects all chords parallel to  $y = mx$  of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , then  $y = mx$  will also bisect all chords parallel to  $y = m'x$ , the common condition being  $mm' = -\frac{b^2}{a^2}$ .

Hence, if a diameter of an ellipse bisects all chords parallel to another diameter, the latter will also bisect all chords parallel to the former.

Two such diameters are referred to as **conjugate diameters** of the ellipse.

## Examples VII

1. (i) Find out the eccentricity, and the co-ordinates of the foci of the ellipse  $9x^2 + 25y^2 = 225$ . [ H. S. 1960 ]

(ii) Find the co-ordinates of the foci of the ellipse  $9x^2 + 5y^2 = 45$ .

2. An ellipse has its major axis along the  $x$ -axis and minor axis along the  $y$ -axis. Its eccentricity is  $\frac{1}{2}$  and the distance between the foci is 4. Find its equation and show that the ellipse passes through the point (2, 3). [ H. S. 1961, Compartmental ]

[ Let the equation of the ellipse be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

Distance between the foci =  $2ae = 4$  or  $2a \cdot \frac{1}{2} = 4$  [since  $e = \frac{1}{2}$ ]  $\therefore a = 4$ .

Again  $b^2 = a^2(1 - e^2) = 16(1 - \frac{1}{4}) = 12$ .

Hence the equation of the ellipse is  $\frac{x^2}{16} + \frac{y^2}{12} = 1$  which is satisfied by (2, 3).]

3. (i) Find the equation to the ellipse whose centre is the origin, whose axes are the axes of co-ordinates, and which passes through the points  $(-3, \frac{16}{5})$  and  $(0, -4)$ .

Find also the co-ordinates of its foci.

[ Let the equation of the ellipse be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

Since it passes through  $(-3, \frac{16}{5})$  and  $(0, -4)$ ,

$$\frac{9}{a^2} + \frac{256}{25b^2} = 1 \quad \text{and} \quad \frac{16}{b^2} = 1.$$

Solving,  $a^2 = 25$  and  $b^2 = 16$ .

Hence the equation of the ellipse is  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ .

Again  $16 = 25(1 - e^2)$  or  $e = \pm \frac{3}{5}$ .

$\therefore$  co-ordinates of foci  $(\pm ae, 0)$  i.e.,  $(\pm 5 \cdot \frac{3}{5}, 0)$  i.e.,  $(\pm 3, 0)$ .]

(ii) An ellipse having centre as origin and axes along the co-ordinate axes, passes through the points  $(\frac{3}{2}, -3)$  and  $(-\sqrt{6}, 2)$ . Find the equations to its directrices.

4. Find the equation to the ellipse having centre as origin, and axes along the axes of co-ordinates, whose latus rectum is 6 and eccentricity  $\frac{1}{2}$ . Write down the co-ordinates of the extremities of its minor axis.

5. (i) The latus rectum of an ellipse is half its major axis; find its eccentricity.

(ii) The distance between the focus and directrix of an ellipse is 16 inches and its eccentricity is  $\frac{3}{4}$ . Obtain the lengths of its principal axes.

6. Find the equation to the ellipse whose focus is  $(-1, 1)$ , eccentricity is  $\frac{1}{2}$ , and directrix is  $x - y + 3 = 0$ .

7. Find the latus rectum, eccentricity and co-ordinates of the centre and foci of the ellipse :

(i)  $3x^2 + 4y^2 + 6x - 8y = 5$ .

(ii)  $9x^2 + 5y^2 - 30y = 0$ .

8. Is the point (i)  $(2, -1\frac{1}{2})$ , (ii)  $(2, -\frac{1}{2})$ , inside or outside the ellipse  $4x^2 + 9y^2 = 36$  ?

9. Find the equation to the tangent of the ellipse  $9x^2 + 16y^2 = 144$  having equal positive intercepts on the axes.

[ H. S. 1961 ]

10. Find the distance from the origin of the point where the tangent at the extremity of a latus rectum of the ellipse  $9x^2 + 25y^2 = 225$  intersects the major axis.

[ H. S. 1960 ]

11. Show that  $x - 3y = 13$  touches the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1. \quad [ H. S. 1960, \text{Compartmental} ]$$

What are the co-ordinates of the point of contact ?

[ Since  $\sqrt{25(\frac{1}{3})^2 + 16} = \frac{13}{3}$  which is equal to the numerical value of the constant of line  $y = \frac{x}{3} - \frac{13}{3}$ . [ Cor. § 7.6 ], hence the line touches the ellipse.

Let  $(x_1, y_1)$  be the point of contact.

The equation of tangent at  $(x_1, y_1)$

$$\frac{xx_1}{25} + \frac{yy_1}{16} = 1 \text{ must be identical with } x - 3y = 13.$$

$$\frac{x_1/25}{1} = \frac{y_1/16}{-3} = \frac{1}{13}. \therefore \text{ co-ordinates of point of contact } \left( \frac{25}{13}, -\frac{48}{13} \right).$$

12. Find the equations to the tangents to the ellipse  $9x^2 + 16y^2 = 36$  which are parallel to  $3x - 3y + 7 = 0$ , and find out the points of contact.

13. If a tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  intercepts lengths  $\alpha$  and  $\beta$  along the axes, prove that  $a^2/\alpha^2 + b^2/\beta^2 = 1$ .

[ Equation of tangent is  $\frac{x}{a} + \frac{y}{b} = 1$ .

$\therefore$  since tangent to  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .  $\therefore \beta = \sqrt{a^2 \left(-\frac{\beta}{a}\right)^2 + b^2}$

or,  $\frac{a^2}{a^2} + \frac{b^2}{\beta^2} = 1$ . ]

14. Prove that the product of the perpendiculars from the foci on any tangent to an ellipse is constant, and is equal to the square on the semi-minor axis.

15. The straight line  $3x - 5y + 25 = 0$  touches an ellipse whose principal axes are along the axes of co-ordinates, and whose eccentricity is given to  $\frac{3}{5}$ . Find the distance between the foci of the ellipse.

[ Let the equation of ellipse be  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

$\therefore b^2 = a^2(1 - e^2) = a^2(1 - \frac{9}{25}) = \frac{16}{25}a^2$ . ... (1)

Since  $3x - 5y + 25 = 0$  is a tangent to the ellipse,

$\therefore 5 = \sqrt{a^2 \left(\frac{3}{5}\right)^2 + b^2}$  or,  $25 = \frac{9}{25}a^2 + b^2$ . ... (2)

Solving (1) and (2),  $a = 5$ .

Co-ordinates of foci  $(\pm ae, 0)$  i.e.,  $(\pm 3, 0)$ .

Distance between foci  $= 2ae = 6$ . ]

16. Find out the equation to the normal to the ellipse  $2x^2 + 7y^2 = 71$  at  $(2, -3)$  and determine the distance of the point where it intersects the major axis, from the foot of the ordinate.

[ Equation of ellipse is  $\frac{x^2}{\frac{71}{2}} + \frac{y^2}{\frac{71}{7}} = 1$ .

As in § 7.5, equation of normal is  $\frac{x - 2}{\frac{71}{2}} = \frac{y + 3}{-\frac{71}{7}}$

or  $21x + 4y = 30$ .

When it intersects major axis, i.e.,  $x$ -axis,  $y = 0$ .

$\therefore x = \frac{30}{21}$  at  $(\frac{10}{7}, 0)$ .

Distance of the point  $(\frac{10}{7}, 0)$  from the foot of the ordinate  $= \frac{30}{7} - 2 = -\frac{4}{7}$ . ]

17. Write down the equation to the normal to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at an extremity of the latus rectum, and show that



if it passes through an extremity of the minor axis, the eccentricity of the ellipse is given by  $e^2 = \frac{1}{2}(\sqrt{5} - 1)$ .

18. If the normal to the ellipse  $x^2 + 3y^2 = 12$  at a point be inclined at  $60^\circ$  to the major axis, show that the line joining the centre to the point is inclined at  $30^\circ$  to the same axis.

19. Obtain the equation to the chord of the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  which is bisected at the point  $(2, -1)$ .

20. Find the length of the chord of the ellipse  $\frac{x^2}{25} + \frac{y^2}{16} = 1$  intercepted by the line  $x + y = 3$ . What are the co-ordinates of its middle point?

21. Find the equation to the diameter of the ellipse  $6x^2 + 9y^2 = 1$  bisecting all chords parallel to  $y = x$ .

[As in § 7.9, equation of diameter is  $y = -\frac{b^2}{a^2 m}x$ .

Hence  $b^2 = \frac{1}{3}$ ,  $a^2 = \frac{1}{6}$  and  $m = 1$ .

$\therefore y = -\frac{\frac{1}{3}}{\frac{1}{6}-1}x$ , or  $3y + 2x = 0$ .]

22. Show that the straight lines  $3y = 4x$  and  $x + 3y = 0$  each bisects all chords of the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  parallel to the other.

### ANSWERS

1. (i)  $\frac{4}{3}$ ;  $(\pm 4, 0)$ . (ii)  $(0, \pm 2)$ . 2.  $\frac{x^2}{16} + \frac{y^2}{12} = 1$ .
3. (i)  $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ;  $(\pm 3, 0)$ . (ii)  $y = \pm 4\sqrt{3}$ . 4.  $\frac{x^2}{16} + \frac{y^2}{12} = 1$ ;  $(0, \pm 2\sqrt{3})$ .
5. (i)  $\frac{1}{\sqrt{2}}$ . (ii) 30 inches, 24 inches.
6.  $8\{(x+1)^2 + (y-1)^2\} = (x-y+3)^2$ , or,  $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$ .
7. (i)  $3; \frac{1}{2}; (-1, 1); (0, 1)$  and  $(-2, 1)$ . (ii)  $3\frac{1}{2}; \frac{2}{3}; (0, 3); (0, 1)$  and  $(0, 5)$ .
8. (i) Outside. (ii) Inside. 9.  $x + y = 5$ , 10.  $6\frac{1}{2}$ . 11.  $(\frac{1}{2}, -\frac{1}{2})$ .
12.  $2x - 2y = \pm 5$ ;  $(\frac{3}{2}, -\frac{1}{2})$  and  $(-\frac{3}{2}, \frac{1}{2})$ . 15. 6. 16.  $21x + 4y = 30$ ;  $-4$ .
17.  $x = e(y + ae^2)$ . 19.  $8x - 9y = 25$ . 20.  $7\frac{1}{2}; (\frac{1}{2}, \frac{1}{2})$ . 21.  $2x + 3y = 0$ .

## CHAPTER VIII

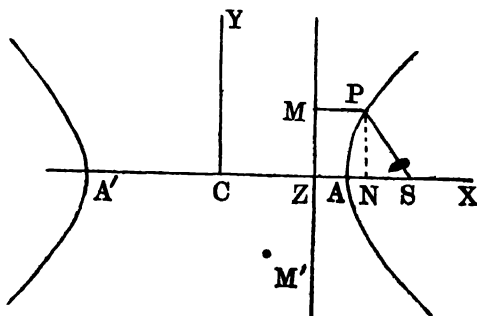
### HYPERBOLA

#### 8'1. Hyperbola.

A *hyperbola* is a curve traced out by a point which moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the ratio being greater than unity.

The fixed point is called the *focus*, the fixed straight line is called the *directrix*, and the constant ratio (which is greater than unity in this case) is called the *eccentricity* of the hyperbola.

#### 8'2. Standard equation of a hyperbola.



Let  $S$  be the focus,  $MM'$  the directrix, and  $e (> 1)$  the given eccentricity of the hyperbola.

Draw  $SZ$  perpendicular from  $S$  on  $MM'$ , and let it be divided internally at  $A$  and externally at  $A'$  in the ratio  $e : 1$ . As  $e > 1$ ,  $SA' > A'Z$ , and accordingly  $A'$  is to the left of  $S$  as in the figure, on the side of the directrix  $MM'$  opposite to  $A$ ,  $S$  being not between  $A$  and  $A'$ .

Then,  $SA = e \cdot AZ$  and  $SA' = e \cdot A'Z$ .

Hence, by definition of the hyperbola,  $A$  and  $A'$  are points on the hyperbola.

Let  $C$  be the middle point of  $AA'$ .

Thus,  $SA + SA' = e(AZ + A'Z)$

and  $SA' - SA = e(A'Z - AZ)$ ,

or,  $2CS = e.AA' = e.2CA$  and  $AA'$  or  $2CA = e.2CZ$ .

Let  $CA = CA' = a$ . Thus,  $CS = ae$ , and  $CZ = \frac{a}{e}$ .

Let us choose  $C$  as origin, and  $CX$  along  $A'A$  as  $x$ -axis, the  $y$ -axis  $CY$  being parallel to  $M'M$  i.e., perpendicular to  $A'A$  through  $C$ .

Now,  $P$  being any point on the hyperbola, whose co-ordinates are  $(x, y)$ , let  $PN$  be the perpendicular from  $P$  to the  $x$ -axis, and  $PM$  be perpendicular to the directrix  $MM'$ . Then,  $CN = x$ ,  $PM = ZN = CN - CZ = x - \frac{a}{e}$ . Also co-ordinates of  $S$  are evidently  $(ae, 0)$ . [ $\because CS = ae$ ].

Hence, from the property of the hyperbola,

$$SP = e.PM, \text{ or, } SP^2 = e^2 PM^2.$$

$$\therefore (x - ae)^2 + y^2 = e^2 \left( x - \frac{a}{e} \right)^2,$$

$$\text{or, } x^2(e^2 - 1) - y^2 = a^2(e^2 - 1), \quad (\because e > 1 \text{ here})$$

$$\text{or, writing } a^2(e^2 - 1) = b^2,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \dots \dots (i)$$

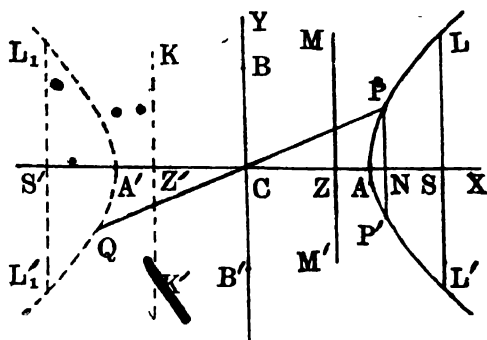
This being the relation satisfied by the co-ordinates of any point on the hyperbola, it represents the equation to the hyperbola in its standard form.

Here  $C$ , the middle point of  $AA'$  (called the *centre*) is the origin,  $CA = CA' = a$ , and  $b^2 = a^2(e^2 - 1)$ .

### 8.3. Shape and elementary properties of the hyperbola.

From the equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  of the hyperbola, the following points may be noted.

If  $y=0$ ,  $x=\pm a$ , so that the hyperbola cuts the  $x$ -axis at points  $A$  and  $A'$  given by  $x=a$  and  $x=-a$  respectively.



If  $x=0$ ,  $y^2$  is negative and so  $y$  is imaginary. Accordingly the curve does not cut the  $y$ -axis at all.

For values of  $x < a$  or  $> -a$  (i.e., within  $AA'$ ),  $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$  is negative, and so  $y$  is imaginary. Hence, there is no portion of the hyperbola within the range  $AA'$ .

For values of  $x > a$  or  $< -a$ ,  $\frac{x^2}{a^2} > 1$  and so  $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$  is positive. Hence,  $y$  has two equal and opposite values. Thus from  $A$  to the right, or from  $A'$  to the left, the curve extends, being symmetrical with respect to the  $x$ -axis,  $y$  having greater and greater magnitudes as the magnitude of  $x$  becomes greater and greater.

Again, for any values of  $y$ ,  $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$  is positive, and so  $x$  has two equal and opposite values. Hence, the curve is symmetrical with respect to the  $y$ -axis.

Thus, the hyperbola consists of two detached portions as shown in the figure, extending from  $A$  towards the right and from  $A'$  toward the left, being symmetrical about both the  $x$ -axis and the  $y$ -axis.

From symmetry about the  $y$ -axis  $CY$ , we see that if we take points  $S'$  and  $Z'$  on the  $x$ -axis such that  $CS' = CS$  and  $CZ' = CZ$

on opposite sides of  $C$ , and draw  $KZ'K'$  parallel to  $MZM'$ , the curve might be drawn equally well with  $S'$  as focus and  $KZ'K'$  as directrix,  $e$  being the same as before. Hence, a hyperbola has a second focus and a second directrix symmetrically situated with respect to  $C$ .

Lastly, if  $(x_1, y_1)$  be the co-ordinates of a point  $P$  on the hyperbola so that they satisfy the equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , the co-ordinates  $(-x_1, -y_1)$  will also satisfy it, and accordingly the diagonally opposite point  $Q$ , where  $PQ$  is bisected at  $C$ , is also a point on the hyperbola. Thus, every chord of the hyperbola through  $C$  is bisected at  $C$ , and so the hyperbola is symmetrical with respect to the origin  $C$ , the mid-point of  $AA'$ . This is why  $C$  is called the centre of the hyperbola.

The  $x$ -axis is here referred to as the *transverse axis*, and the length  $AA' = 2a$  is called the length of the transverse axis. The  $y$ -axis here is referred to as the *conjugate axis*, and a length  $BB' = 2b$  (where  $CB = CB' = b$ ) along this axis is referred to as the length of the conjugate axis.

The chord  $LSL'$  through the focus  $S$  (or  $L_1S'L'_1$  through the focus  $S'$ ) perpendicular to the transverse axis (i.e., parallel to the directrix) is called the *latus rectum* of the hyperbola.

Now,  $ae$  being the length  $CS$ , the  $x$ -co-ordinate of the extremity  $L$  of the latus rectum is  $ae$ . Hence, from the equation to the hyperbola, the  $y$ -co-ordinate of  $L$  is given by

$$\frac{a^2 e^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$\text{Hence, } y = \pm b \sqrt{e^2 - 1} = \pm a(e^2 - 1).$$

Thus, the length  $LL'$  of the latus rectum

$$= 2a(e^2 - 1) = 2 \frac{b^2}{a}.$$

$$\therefore \text{ semi-latus rectum } = \frac{b^2}{a} = a(e^2 - 1).$$

Co-ordinates of the extremity  $L$  of the latus rectum are  $[ae, a(e^2 - 1).]$

The eccentricity of the hyperbola is given by

$$b^2 = a^2(e^2 - 1), \text{ or, } e^2 = \frac{a^2 + b^2}{a^2}.$$

**Note 1.** If  $a=b$ , the hyperbola is said to be a **rectangular or equilateral hyperbola**. For a rectangular hyperbola the eccentricity

$$= \sqrt{2}.$$

**Note 2.** Lengths of the focal distances  $SP, S'P$  of any point  $P$  on the hyperbola.

Let the co-ordinates of  $P$  be  $(x_1, y_1)$ . Those of  $S$  being  $(ae, 0)$  we get

$$\begin{aligned} SP^2 &= (x_1 - ae)^2 + y_1^2 = (x_1 - ae)^2 + b^2 \left( \frac{x_1^2}{a^2} - 1 \right) \\ & \quad \quad \quad [ \text{from the equation to the hyperbola} ] \\ &= (x_1 - ae)^2 + (e^2 - 1)(x_1^2 - a^2) \quad [ \because b^2 = a^2(e^2 - 1) ] \\ &= e^2 x_1^2 - 2x_1 ae + a^2 = (ex_1 - a)^2. \end{aligned}$$

$\therefore SP = ex_1 - a$ , which is the positive value of  $SP$ ,

$\because x_1 > a$  and  $e > 1$  here.

Similarly,  $S'P = ex_1 + a$ .

Thus,  $S'P - SP = 2a = \text{length of the transverse axis}$ .

Hence, we get the important property of the hyperbola, namely, *the difference of the focal distances of any point on the hyperbola is constant and equal to the transverse axis*.

**8'4. Equation to the tangent at a given point  $(x_1, y_1)$  on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .**

Let  $P$  be the point  $(x_1, y_1)$  on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots \dots \dots \text{ (i)}$$

and let  $Q(x_2, y_2)$  be a neighbouring point on it. The equation to the chord  $PQ$  is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad \dots \dots \text{ (ii)}$$

Now, since  $P$  and  $Q$  both lie on the hyperbola (i), we have

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \dots \text{ (iii)} \quad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1 \quad \dots \text{ (iv)}$$

Hence, subtracting,

$$\frac{x_2^2 - x_1^2}{a^2} - \frac{y_2^2 - y_1^2}{b^2} = 0, \text{ or, } \frac{y_2 - y_1}{x_2 - x_1} = \frac{b^2}{a^2} \cdot \frac{x_2 + x_1}{y_2 + y_1}.$$

∴ equation (ii) can be written as

$$y - y_1 = \frac{b^2}{a^2} \cdot \frac{x_2 + x_1}{y_2 + y_1} (x - x_1) \quad \dots (v)$$

Now, make  $Q$  approach  $P$  and ultimately coincide with it, so that the co-ordinates  $(x_2, y_2)$  coincide with  $(x_1, y_1)$ . In the limiting position the straight line  $PQ$  becomes the tangent at  $P$  whose equation [from (v)] then becomes

$$y - y_1 = \frac{b^2}{a^2} \cdot \frac{x_1}{y_1} (x - x_1),$$

$$\text{or, } \frac{y_1}{b^2} (y - y_1) = \frac{x_1}{a^2} (x - x_1),$$

$$\text{or, } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1. \quad [\text{by (ii)}]$$

Hence, the equation to the tangent at  $(x_1, y_1)$  to the hyperbola (i) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

**8.5. Equation to the normal at  $(x_1, y_1)$  to the hyperbola**  
 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

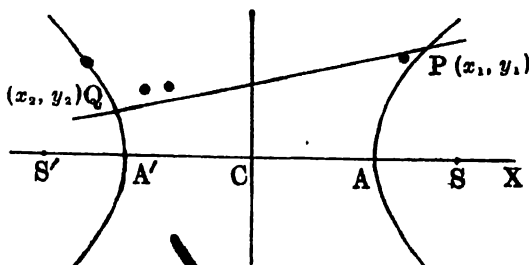
The tangent at  $(x_1, y_1)$  to the hyperbola is  $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$   
 or,  $y = \frac{b^2 x_1}{a^2 y_1} x - \frac{b^2}{y_1}$ , of which the 'm' is  $\frac{b^2 x_1}{a^2 y_1}$ .

The normal which is perpendicular to the tangent through  $(x_1, y_1)$ , has its 'm' =  $-\frac{a^2 y_1}{b^2 x_1}$ , and accordingly its equation is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1} (x - x_1),$$

$$\text{or, } \frac{x - x_1}{a^2} = \frac{y - y_1}{b^2}.$$

8'6. Length of the chord of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , intercepted by the straight line  $y = mx + c$ .



At the points of intersection of the line with the hyperbola, both the equations are satisfied. Hence, eliminating  $y$  between the two equations, the abscissæ of the points of intersection will be given by

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1,$$

$$\text{or, } (a^2 m^2 - b^2)x^2 + 2mca^2 x + a^2(b^2 - c^2) = 0 \quad (i)$$

which being a quadratic in  $x$ , there are only two values of  $x$  and accordingly only two points of intersection of the given straight line with the hyperbola (real, coincident or imaginary).

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of the two points  $P$  and  $Q$  of intersection. Then,  $x_1$  and  $x_2$  are the roots of (i).

$$\therefore x_1 + x_2 = -\frac{2mca^2}{a^2 m^2 - b^2} \text{ and } x_1 x_2 = \frac{a^2(b^2 - c^2)}{a^2 m^2 - b^2}.$$

$$\begin{aligned} \therefore (x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1 x_2 \\ &= \frac{4m^2 c^2 a^4}{(a^2 m^2 - b^2)^2} - \frac{4a^2(b^2 - c^2)}{a^2 m^2 - b^2} \\ &= \frac{4a^2 \{m^2 c^2 a^2 - (b^2 - c^2)(a^2 m^2 - b^2)\}}{(a^2 m^2 - b^2)^2} \\ &= \frac{4a^2 b^2 (c^2 - a^2 m^2 + b^2)}{(a^2 m^2 - b^2)^2}. \end{aligned}$$



Again,  $P$  and  $Q$  lying on the given line  $y = mx + c$ ,

$$y_1 = mx_1 + c, \quad y_2 = mx_2 + c. \quad \therefore y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$  length of the chord  $PQ$

$$\begin{aligned} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + m^2)} \\ &= \sqrt{\frac{4a^2b^2(c^2 - a^2m^2 + b^2)(1 + m^2)}{(a^2m^2 - b^2)^2}} \\ &= \frac{2ab\sqrt{1+m^2}\sqrt{c^2 - a^2m^2 + b^2}}{a^2m^2 - b^2}. \end{aligned}$$

**Cor. Condition of tangency.**

The given line will touch the hyperbola only when the two points of intersection come into coincidence, i.e., when the length of the chord intercepted is zero. Hence, the condition that the given line  $y = mx + c$  may touch the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is

$$c^2 - a^2m^2 + b^2 = 0, \text{ or, } c = \pm \sqrt{a^2m^2 - b^2}.$$

**87.** To show that  $y = mx + \sqrt{a^2m^2 - b^2}$  is a tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  for all values of  $m$ , and to find the point of contact.

The tangent at  $(x_1, y_1)$  of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is  $\frac{xx_1}{a^2}$

$$- \frac{yy_1}{b^2} = 1, \quad \text{or, } y = \frac{b^2x_1}{a^2y_1}x - \frac{b^2}{y_1}. \quad \dots (i)$$

If the line  $y = mx + \sqrt{a^2m^2 - b^2}$  ... (ii) be a tangent to the hyperbola at  $(x_1, y_1)$ , the equations (i) and (ii) must be identical. Hence, comparing coefficients,

$$\frac{b^2x_1}{a^2y_1} = m, \quad -\frac{b^2}{y_1} = \sqrt{a^2m^2 - b^2};$$

$$y_1 = -\frac{b^2}{\sqrt{a^2m^2 - b^2}}, \quad x_1 = \frac{ma^2y_1}{b^2} = -\frac{ma^2}{\sqrt{a^2m^2 - b^2}}.$$

The line (ii) therefore will touch the hyperbola only if the assumed point  $(x_1, y_1)$  is really a point on the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , i.e., if

$$\left( -\frac{am}{\sqrt{a^2m^2 - b^2}} \right)^2 - \left( \frac{-b}{\sqrt{a^2m^2 - b^2}} \right)^2 = 1$$

which is evidently satisfied.

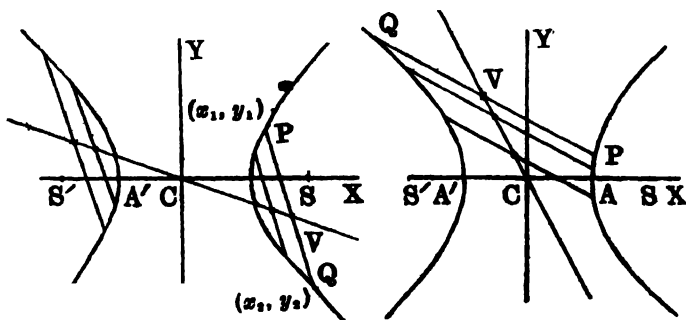
Thus,  $y = mx + \sqrt{a^2m^2 - b^2}$  is always a tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , whatever  $m$  may be, and the point of contact is given by

$$x_1 = -\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \quad y_1 = -\frac{b^2}{\sqrt{a^2m^2 - b^2}}$$

Similarly,  $y = mx - \sqrt{a^2m^2 - b^2}$  is also a tangent to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , for all values of  $m$ , the co-ordinates of the point of contact being

$$\left( \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}} \right).$$

8.8. Locus of the middle points of a system of parallel chords ; diameter.



Let  $PQ$ , given by the equation

$$y = mx + c \quad \dots$$

be any one of a system of parallel chords of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \dots \text{ (ii)}$$

As the chords are parallel,  $m$  is the same for all chords, but  $c$  is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating  $y$ , the abscissæ are given by the roots of the equation

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1,$$

$$\text{or, } (a^2 m^2 - b^2) x^2 + 2a^2 m c x + a^2 (b^2 + c^2) = 0. \quad \dots \text{ (iii)}$$

Thus, if  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of  $P$  and  $Q$ ,  $x_1, x_2$  are the roots of (iii), and so.

$$x_1 + x_2 = -\frac{2a^2 m c}{a^2 m^2 - b^2}.$$

Hence, if  $(X, Y)$  be the co-ordinates of the mid-point  $V$  of  $PQ$ ,

$$X = \frac{1}{2}(x_1 + x_2) = -\frac{a^2 m c}{a^2 m^2 - b^2}.$$

Also,  $\therefore V$  is a point on (i),  $Y = mX + c$ .

$\therefore$  eliminating  $c$ ,

$$X = -\frac{a^2 m (Y - mX)}{a^2 m^2 - b^2}, \text{ or, } -b^2 X = -a^2 m Y,$$

$$\text{or, } Y = \frac{b^2}{a^2 m} X, \text{ which being independent of } c \text{ holds for}$$

the middle point of any chord of the parallel system. Hence, the locus of the middle points of a system of parallel chords of the hyperbola, parallel to  $y = mx$ , is

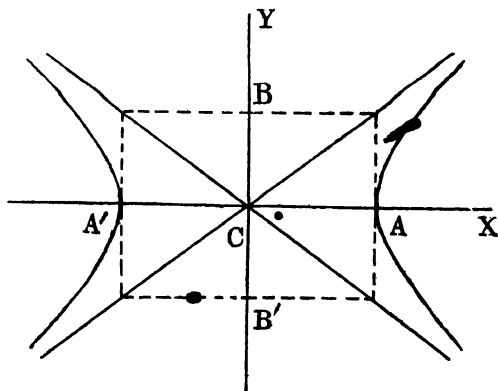
$$y = \frac{b^2}{a^2 m} x$$

which is evidently a straight line passing through the origin *i.e.*, the centre  $C$  of the hyperbola. This straight line is called a *diameter* of the hyperbola. For different values of  $m$  (*i.e.*, for differently directed system of parallel chords) we get different diameters, all passing through the centre.

### 8.9. Asymptotes of a hyperbola.

We have noticed in § 8.7 that the line given by  $y = mx + \sqrt{a^2 m^2 - b^2}$  is always a tangent\* to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , and the co-ordinates of the point of contact are  $\left( -\frac{a^2 m}{\sqrt{a^2 m^2 - b^2}}, -\frac{b^2}{\sqrt{a^2 m^2 - b^2}} \right)$ . Now, if  $m$  be so chosen that  $a^2 m^2 - b^2 = 0$ , or,  $m = \pm \frac{b}{a}$ , the co-ordinates of the point of contact both tend to infinity.

Hence, the straight lines  $y = \pm \frac{b}{a} x$  are both tangents to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , where the points of contact tend to infinity. These lines are defined as *asymptotes* to the hyperbola.



They are inclined to the transverse axis at an angle  $\theta$ , where  $\tan \theta = \pm \frac{b}{a}$ . Hence, with sides equal and parallel to the transverse axis  $2a$  and conjugate axis  $2b$  of the hyperbola, if we construct a rectangle with centre at the origin, the diagonals will be the asymptotes, which will continually approach the hyperbola, and will ultimately touch it at infinite distance.

In the particular case when  $b = a$ , the asymptotes are inclined to the  $x$ -axis at angles  $\pm 45^\circ$ , and so they are mutually perpendicular. The hyperbola in this case, when its transverse and conjugate axes are equal in length, is defined to be a *rectangular or equilateral hyperbola*, having its asymptotes mutually perpendicular.

### 8'10. Illustrative Examples.

**Ex. 1.** The co-ordinates of the foci of a hyperbola are  $(-5, 3)$  and  $(7, 3)$ , and its eccentricity is  $\frac{5}{4}$ . Find the equation and determine the length of its latus rectum.

Let  $S(7, 3)$  and  $S'(-5, 3)$  be the given foci. The eccentricity  $e = \frac{5}{4}$ . If  $2a$  be the length of the transverse axis, then  $SS' = 2ae$ ,

$$\text{or, } 12 = 2a \times \frac{5}{4}. \quad \therefore a = 4.$$

Also,  $2b$  being the conjugate axis,

$$b^2 = a^2(e^2 - 1) = 16\left(\frac{25}{16} - 1\right) = 20.$$

Hence, the length of the latus rectum

$$= 2 \frac{b^2}{a} = 2 \cdot \frac{10}{4} = 10.$$

Again, the middle point  $C$  of  $SS'$  is clearly the centre, and its co-ordinates are

$$\frac{1}{2}(7-5) \text{ and } \frac{1}{2}(3+3), \text{ i.e., } (1, 3).$$

Also, the transverse axis, which is along the line  $SS'$  has equation

$$(y-3)(7+5) = (x-7)(3-3) = 0, \text{ i.e., } y = 3$$

and hence it is parallel to the  $x$ -axis.

Now referred to the centre  $C$  as origin, and transverse axis as  $x$ -axis, the equation to the hyperbola (whose  $a^2 = 16$  and  $b^2 = 20$ ) is evidently

$$\frac{x^2}{16} - \frac{y^2}{20} = 1.$$

Hence, referred to given axes [ with reference to which  $C$  has co-ordinates  $(1, 3)$ , and to which the transverse and conjugate axes of the hyperbola are parallel ] the required equation to the hyperbola is evidently

$$\frac{(x-1)^2}{16} - \frac{(y-3)^2}{20} = 1. \quad \dots \quad \dots \quad (i)$$

*Alternatively*

Since the difference of the focal distances of any point on the hyperbola is equal to its transverse axis, which is  $2a=8$  here, if  $(x, y)$  be the co-ordinates of any point on the hyperbola,

$$\sqrt{(x+5)^2 + (y-3)^2} - \sqrt{(x-7)^2 + (y-3)^2} = \pm 8.$$

or,  $\sqrt{(x+5)^2 + (y-3)^2} = \sqrt{(x-7)^2 + (y-3)^2} \pm 8.$

Hence, squaring and transposing,

$$24x - 88 = \pm 16 \sqrt{(x-7)^2 + (y-3)^2},$$

or,  $(3x - 11)^2 = 4\{(x-7)^2 + (y-3)^2\},$

or,  $5x^2 - 4y^2 - 10x + 24y - 111 = 0$

which is the required equation to the hyperbola, and is the same as equation (i) already obtained above.

**Ex. 2.** Prove that the tangent to the hyperbola  $x^2 - 3y^2 = 12$  at the point  $(-6, 2\sqrt{2})$  bisects the angle between the focal distances of the point.

The given equation to the hyperbola can be written in the form

$$\frac{x^2}{12} - \frac{y^2}{4} = 1 \quad \dots \quad \dots \quad \dots \quad (i)$$

and hence the co-ordinates of its foci  $S$  and  $S'$  are easily seen to be

$$(\pm \sqrt{12+4}, 0) \text{ i.e., } (\pm 4, 0).$$

Now  $P$  being the point  $(-6, 2\sqrt{2})$  on the hyperbola, the equations to the focal distances  $SP$  and  $S'P$  are

$$y = \frac{2\sqrt{2}}{-6-4}(x-4), \text{ i.e., } x\sqrt{2} + 5y - 4\sqrt{2} = 0 \quad \dots \quad (ii)$$

and  $y = \frac{2\sqrt{2}}{-6+4}(x+4), \text{ i.e., } x\sqrt{2} + y + 4\sqrt{2} = 0 \quad \dots \quad (iii)$

respectively.

The equation to the bisector of the angle  $SPS'$ , i.e., between (ii) and (iii), in which the origin lies, is

$$\frac{x\sqrt{2} + 5y - 4\sqrt{2}}{\sqrt{2+25}} = \frac{x\sqrt{2} + y + 4\sqrt{2}}{\sqrt{2+1}}$$

or,  $x\sqrt{2} + 5y - 4\sqrt{2} + 3(x\sqrt{2} + y + 4\sqrt{2}) = 0,$

or,  $x + \sqrt{2}y + 2 = 0. \quad \dots \quad \dots \quad (iv)$

Now, equation to the tangent to (i) at  $(-6, 2\sqrt{2})$  is

$$\frac{x(-6)}{12} - \frac{y(2\sqrt{2})}{4} = 1,$$

or,  $x + \sqrt{2}y + 2 = 0$ , the same as (iv). Hence, the tangent at the point is the bisector of the angle between the focal distances of the point.

**Ex. 3.** Find the length of the chord of the hyperbola  $x^2 - 4y^2 = 9$  along the straight line  $x + 4y + 3 = 0$ , and determine the co-ordinates of its middle point.

At the points of intersection of the hyperbola  $x^2 - 4y^2 = 9$  ... (i) with the straight line  $x + 4y + 3 = 0$  ... (ii), eliminating  $x$  the ordinates are the roots of

$$(4y + 3)^2 - 4y^2 = 9, \text{ or, } y(y + 2) = 0.$$

$\therefore y = 0$ , or  $-2$ . The corresponding values of  $x$  from (ii) are  $x = -3$  or  $5$

Hence, the co-ordinates of the extremities of the chord are  $(-3, 0)$  and  $(5, -2)$ .

Thus, the length of the chord  $= \sqrt{(-3-5)^2 + (0+2)^2} = 2\sqrt{17}$ .

Also, the co-ordinates of the middle point of the chord are  $[\frac{1}{2}(-3+5), \frac{1}{2}(0-2)]$  i.e.,  $(1, -1)$ .

**Ex. 4.** Prove that the portion of the tangent at any point of a hyperbola intercepted between the asymptotes is bisected at the point of contact.

Let  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  ..... (i) be the equation to a hyperbola. Its asymptotes are given by  $y = \frac{b}{a}x$  ..... (ii) and  $y = -\frac{b}{a}x$  ..... (iii).

The tangent at any point  $P(x', y')$  to the hyperbola (i) is  $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$  ... (iv). This meets (ii) at a point  $Q$  whose  $x$ -co-ordinate [ by eliminating  $y$  between (ii) and (iv) ] is given by

$$\frac{xx'}{a^2} - \frac{y'}{b^2} \cdot \frac{b}{a}x = 1, \text{ or } x = \frac{a}{\frac{x'}{a} - \frac{y'}{b}}.$$

Similarly (iv) meets (iii) at  $R$  whose  $x$ -co-ordinate is given by

$$x = \frac{a}{\frac{x'}{a} + \frac{y'}{b}}.$$

The  $x$ -co-ordinate of the mid-point of  $QR$  is

$$\frac{1}{2} \left[ \frac{a}{\frac{x'}{a} - \frac{y'}{b}} + \frac{a}{\frac{x'}{a} + \frac{y'}{b}} \right] = \frac{x'}{\frac{x'^2}{a^2} - \frac{y'^2}{b^2}} = x \text{ [ by (i) ].}$$

Similarly, the  $y$ -co-ordinate of the mid-point of  $QR$  is  $y'$ . Thus,  $P$  is the mid-point of  $QR$ .

## Exam 11

1. Obtain the equation to the hyperbola whose focus is  $(a, 0)$ , directrix is the straight line  $x = \frac{1}{2}a$ , and eccentricity is  $\sqrt{2}$ .

[ H. S. 1960 ]

2. Find the equation to the hyperbola, referred to its axes as axes of co-ordinates.

(i) whose eccentricity is  $\sqrt{2}$ , and distance between its foci 16.

(ii) whose latus rectum is  $10\frac{2}{3}$  and distance between focus and directrix is  $3\frac{1}{3}$ .

3. In the hyperbola  $4x^2 - 9y^2 = 36$ , find the lengths of the axes, the co-ordinates of the foci, the eccentricity and the length of the latus rectum.

[ H. S. 1961 ]

[ Equation of hyperbola is  $\frac{x^2}{9} - \frac{y^2}{4} = 1$ .

Comparing with the standard form  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , we have

$$\text{major-axis} = 2a = 2 \cdot 3 = 6$$

$$\text{minor-axis} = 2b = 2 \cdot 2 = 4.$$

$$\text{Again } b^2 = a^2(e^2 - 1), \text{ or } 4 = 9(e^2 - 1). \therefore e = \frac{\sqrt{13}}{3}.$$

Co-ordinates of foci are  $(\pm ae, 0)$ , i.e.,  $(\pm$

$$\text{Latus-rectum} = \frac{2b^2}{a} = \frac{8}{3}.]$$

4. A point moves on the plane of the co-ordinate axes so that the difference of its distances from the points  $(\pm 3, 0)$  is always 4. Prove that it traces out a hyperbola whose eccentricity and length of latus rectum you are to determine.

5. By transferring the origin suitably, show that the equation  $5x^2 - 4y^2 - 20x - 8y - 4 = 0$  represents a hyperbola, and determine its eccentricity, co-ordinates of its foci, and equations to the directrices.

6. Find the co-ordinates of the foci of the hyperbola  $x^2 - y^2 = 9$ .

Also find the distance from the origin of the point where the tangent to the above hyperbola at  $(5, 4)$  meets the  $x$ -axis.

[ H. S, 1960, *Compartmental* ]



[ Equation of hyperbola is  $\frac{x^2}{9} - \frac{y^2}{9} = 1$ .

$$\therefore 9 = 9(e^2 - 1) \quad \therefore e = \sqrt{2}.$$

Co-ordinates of foci  $(\pm 3, \sqrt{2}, 0)$ .

Equation of tangent at  $(5, 4)$  is [ as in § 8.4 ].

$$5x - 4y = 9.$$

When it meets  $x$ -axis,  $y = 0$  i.e.,  $5x = 9$ ,  $\therefore x = \frac{9}{5}$ . ]

7. Show that the tangent to the hyperbola  $\frac{x^2}{16} - \frac{y^2}{9} = 1$  at each of the points (i)  $(-5, \frac{9}{4})$ , (ii)  $(8, 3\sqrt{3})$  bisects the angle between the focal distances of the corresponding point.

8. Find the length intercepted on the conjugate axis between the tangents at the two extremities of a latus rectum of the hyperbola  $7x^2 - 9y^2 = 63$ .

9. (i) Find the points on the hyperbola  $3x^2 - 5y^2 = 15$  at which the tangents are inclined at  $60^\circ$  to the  $x$ -axis.

(ii) Find the tangents perpendicular to  $x + 2y = 0$  of the hyperbola  $7x^2 - 4y^2 = 28$ , and find the points of contact.

10. Prove that the locus of the point of intersection of any two perpendicular tangents to a hyperbola is a circle.

11. Find the equation to the normal to the hyperbola  $16x^2 - 25y^2 = 31$  at the point whose ordinate is  $-3$  and abscissa positive.

12. In the rectangular hyperbola  $x^2 - y^2 = a^2$ , show that

(i) the intercept on the  $x$ -axis of the normal at any point is double the abscissa of the point.

(ii) the length of the normal at any point intercepted between the axes is bisected at the point.

13. Obtain the length of the chord of the hyperbola  $\frac{x^2}{9} - \frac{y^2}{25} = 1$ , passing through the origin and making equal angles with the axes.  
[ H. S. 1960, Compartmental ]

[ Equation of line through origin making equal angles with axes is  $y = x$ .

Solving  $y = x$  and  $25x^2 - 9y^2 = 225$ , let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the points of intersections. Then  $(x_1, x_2) = \pm \frac{1}{4}$

$$(y_1, y_2) = \pm \frac{1}{4}.$$

$$\therefore \text{Length of chord} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \frac{225}{4} + \frac{225}{4} = \frac{15}{\sqrt{2}}.]$$

14. Find the equation to the chord of the hyperbola  $x^2 - 2y^2 = 1$  which is bisected at the point  $(-3, -1)$ .

[ Let the equation of chord be  $y' = mx + c$ .

Since bisected at  $(-3, -1)$ ,

$$\therefore -1 = \frac{1}{1+m} (-3) \quad [\text{as in § 8'8}]$$

$$\therefore m = \frac{2}{3}.$$

Equation of chord becomes  $y = \frac{2}{3}x + c$ .

Since chord passes through  $(-3, -1)$ ,

$$\therefore -1 = \frac{2}{3}(-3) + c. \quad \therefore c = \frac{7}{3}.$$

Hence equation of chord is  $y = \frac{2}{3}x + \frac{7}{3}$  or  $2y = 3x + 7$ . ]

15. Find the length of the chord of the hyperbola  $\frac{x^2}{16} - \frac{y^2}{9} = 1$  along the line  $3x + 2y = 12$ .

16. Find the equation to the diameter of the hyperbola  $\frac{x^2}{4} - \frac{y^2}{5} = 1$  bisecting all chords parallel to  $x - 2y + 7 = 0$ .

17. If  $P$  be a point on a rectangular hyperbola, prove that

$$SP \cdot S'P = CP^2.$$

[ Let  $P(x_1, y_1)$  be any point on hyperbola  $x^2 - y^2 = a^2$ , and  $S$  and  $S'$  are two foci of the hyperbola.

$SP = e \cdot PM$  where  $PM$  is the length of the perpendicular from  $P$  on the directrix

$$x = \frac{a}{e} = e \cdot \frac{x_1 - \frac{a}{e}}{\sqrt{1}} = e \cdot \left(x_1 - \frac{a}{e}\right)$$

Similarly  $S'P = e \left( x_1 + \frac{a}{e} \right)$ .

$$\begin{aligned} \therefore SP \cdot S'P &= e^2 \left( x_1^2 - \frac{a^2}{e^2} \right) = 2 \left( x_1^2 - \frac{a^2}{2} \right) \\ &= 2x_1^2 - a^2 \quad [\text{since } e^2 = 2 \text{ for rectangular hyperbola}] \\ &= 2x_1^2 - (x_1^2 - y_1^2)^2 \quad [\text{since } x_1^2 - y_1^2 = a^2] \\ &= (x_1^2 + y_1^2) \\ &= CP^2. \end{aligned}$$

18. The normal at any point of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  meets the axes in  $M$  and  $N$ , and lines  $MP$  and  $NP$  are drawn at right angles to the axes; prove that the locus of  $P$  is the hyperbola

$$a^2 x^2 - b^2 y^2 = (a^2 + b^2)^2.$$

### ANSWERS

1.  $2x^2 - 2y^2 = a^2$ .
2. (i)  $x^2 - y^2 = 32$ . (ii)  $\frac{x^2}{9} - \frac{y^2}{16} = 1$ .
3. 6, 4;  $(\pm \sqrt{13}, 0)$ ;  $\frac{1}{2} \sqrt{13}$ ;  $2\frac{3}{2}$ .
4.  $\frac{4}{3}$ ; 5.
5.  $\frac{4}{3}$ ;  $(5, -1)$  and  $(-1, -1)$ ;  $x = 3\frac{1}{2}$  and  $x = \frac{3}{2}$ .
6.  $(\pm 3\sqrt{2}, 0)$ ;  $1\frac{1}{2}$ .
8. 6.
9. (i)  $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$  and  $\left(-\frac{5}{2}, -\frac{\sqrt{3}}{2}\right)$ .
- (ii)  $y = 2x \pm 3$ ;  $\left(\frac{2}{3}, \frac{5}{3}\right)$  and  $\left(-\frac{2}{3}, -\frac{5}{3}\right)$ .
11.  $75x - 64y = 492$ .
13.  $\frac{1}{2}\sqrt{2}$ .
14.  $3x - 2y + 7 = 0$ .
15.  $\frac{4}{3}\sqrt{13}$ .
16.  $5x - 2y = 0$ .

ELECTIVE MATHEMATICS, PAPER II  
HIGHER SECONDARY EXAMINATION PAPERS

Board of Secondary Education, W. B.

1962

GROUP A

1. (a) Prove that in an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle, together with twice the rectangle contained by one of these sides and the projection of the other side on it.

(b) Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

2. (a) If two chords of a circle intersect inside the circle, prove that the rectangle contained by the parts of one, is equal to the rectangle contained by the parts of the other.

(b) Through any point  $X$  on the common chord of two intersecting circles, chords  $AB$  and  $CD$  are drawn one in each circle. Prove that  $AX.XB = CX.XD$ .

3. (a) Prove that if two triangles are equiangular their corresponding sides are proportional.

(b) In the trapezium  $ABCD$ ,  $AB$  is parallel to  $DC$ , and the diagonals intersect at  $O$ . Show that  $OA : OC = OB : OD$ .

4. (a) Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(b)  $AD$  is a median of the triangle  $ABC$ , and the angles  $ADB$ ,  $ADC$  are bisected by lines which meet  $AB$ ,  $AC$  at  $E$  and  $F$  respectively. Show that  $EF$  is parallel to  $BC$ .

5. Construct a regular hexagon circumscribing a circle of radius 1.5 inches. Measure a side of the hexagon.

[ Statement of construction as well as justification, are to be given. ]

GROUP B

6. (a) Find the co-ordinates of the point which divides in a given ratio  $m_1 : m_2$  internally, the line joining two given points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

(b) The co-ordinates of the vertices of a triangle are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ . Find co-ordinates of the point where the medians of the triangle intersect.

7. (a) Find the angle between the straight lines whose equations are  $y = m_1x + c_1$  and  $y = m_2x + c_2$ .

(b) Find the equation of the straight line passing through the point  $(-3, 1)$  and perpendicular to the line  $5x - 2y + 7 = 0$ .

8. (a) Find the equation of the circle passing through the origin which makes intercepts 6 and 8 on the positive sides of the axes of  $x$  and  $y$  respectively.

(b) Prove that the centres of the three circles

$$x^2 + y^2 - 2x + 6y = -1$$

$$x^2 + y^2 + 4x - 12y = 9$$

$$\text{and } x^2 + y^2 - 16 = 0$$

lie on a straight line.

9. (a) Find the equation of the parabola, whose focus is at the point  $(5, 0)$  and whose directrix is the line  $3x - 4y + 2 = 0$ .

(a) Show that the straight line  $y = mx + \frac{a}{m}$  is a tangent to the parabola  $y^2 = 4ax$ .

10. (a) Find the equation of the ellipse whose major and minor axes lie along the axes of co-ordinates  $OX$ ,  $OY$  respectively and whose eccentricity is  $\frac{1}{\sqrt{2}}$  and latus rectum is 3.

(b) Show that the line  $x - y = 5$  touches the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

#### GROUP C

11. Prove that all straight lines drawn perpendicular to a given straight line at a given point are coplanar.

12. If a right angle rotates about one of its arms, prove that the other arm describes a plane.

13. Find the volume and the lateral surface of a right prism 8 inches long, standing on an isosceles triangle, each of whose equal sides is 5 inches and the other side 6 inches.

14. A right pyramid stands on a rectangular base whose sides are 12 inches and 9 inches ; and the length of each of the slant edges is 8·5 inches. Find the height and the volume of the pyramid.

## 1963

## GROUP A

1. (a) If two triangles have their sides proportional, when taken in order, prove that they are equiangular.

(b) Prove that the areas of two similar triangles are proportional to the squares on their circum-radii.

2. (a) If the base of a triangle be divided externally in the ratio of the other two sides, prove that the line joining the vertex to this point of division bisects the vertical angle externally.

(b) Prove that the external bisectors of two angles and the internal bisector of the third angle are concurrent.

3. (a) Show that the acute angle made by a tangent to a circle with a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle.

(b) Two circles intersect at  $A$  and  $B$ , and through  $P$ , any point on one of them, straight lines  $PAC$  and  $PBD$  are drawn to cut the other at  $C$  and  $D$ . Show that  $CD$  is parallel to the tangent at  $P$ .

4. Construct, to the scale, an isosceles triangle with each of the equal sides equal to 2 inches, and each base angle double the vertical angle.

Or,

Divide a straight line of length 2 inches into two parts, such that the square on one part may be three times the square on the other.

[ *Statement of construction and full neat traces are to be given in any one of the above cases, but no proof.* ]

## GROUP B

5. (a) Obtain the distance between two points whose rectangular Cartesian co-ordinates are  $(x_1, y_1)$  and  $(x_2, y_2)$ .

(b) Prove that three times the sum of the squares on the side of a the successive angular points of a rectangle.

6. (a) Obtain the perpendicular distance from the point  $(x_1, y_1)$  to the straight line  $ax+by+c=0$ .

(b) Find the orthocentre of the triangle whose angular points are  $(2, 7)$ ,  $(-6, 1)$  and  $(4, -5)$ .

7. (a) Find the equation to the tangent at  $(x_1, y_1)$  of the circle

$$x^2 + y^2 = a^2.$$

(b) Obtain the equation to the circle which passes through the point  $O, 4)$  and touches the  $x$ -axis at the point  $(2, 0)$ .

8. (a) A tangent to the parabola  $y^2=12x$  makes an angle  $45^\circ$  to the axis. Find the co-ordinates of its point of contact.

(b) The co-ordinates of the foci of a hyperbola are  $(5, 0)$ , and  $(-5, 0)$  and its eccentricity is  $\frac{5}{3}$ . Find its equation.

9. (a) Show that the locus of the middle points of a system of parallel chords of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is a straight line passing through its centre.

(b) Find the equation to the normal to the ellipse  $\frac{x^2}{25} + \frac{y^2}{16} = 1$  at an extremity of a latus rectum.

### GROUP C

10. (a) If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane in which they lie.

(b) If  $PA=PB=PC$ , where  $P$  is a point outside the plane of the triangle  $ABC$ , and if  $PO$  be drawn perpendicular to the plane, prove that  $O$  is the circum-centre of the triangle  $ABC$ .

(c) If two straight lines are both perpendicular to a plane, show that they are parallel.

(d) If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

11. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio  $8 : 5$ ; show that the radius of the base is to the height as  $3 : 4$ .

A sphere of diameter 6 cms. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cms. If the sphere be completely submerged, by how much will the surface of the water be raised?

1964

## GROUP A

1. Construct a square equal in area to a given rectangle.

Or, Construct a regular hexagon about a given circle,

(Traces of construction only are required in *either* of the two constructions.)

2. Prove that in every triangle, the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing the angle diminished by twice the rectangle contained by one of those sides and the projection of the other side upon it.

$ABC$  is an isosceles triangle and  $AY$  is drawn to cut the base internally at  $Y$ . Show that  $AY^2 = AB^2 - BY \cdot YC$ .

3. Prove that, if two triangles are equiangular, their corresponding sides are proportional.

Prove that the altitudes of two similar triangles are proportional to the corresponding sides.

4. Prove that if two chords of a circle intersect inside the circle, the rectangle contained by the parts of one is equal to the rectangle contained by the parts of the other.

In a triangle  $ABC$ , perpendiculars  $AP$  and  $BQ$  are drawn from  $A$  and  $B$  to opposite sides and intersect at  $O$ . Prove that

$$AO \cdot OP = BO \cdot OQ.$$

## GROUP B

5. Find the co-ordinates of the point which divides the straight line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  internally in the ratio of  $m : n$ .

Write down the co-ordinates of the middle point of the straight line joining the points  $(7, -4)$  and  $(-5, 6)$ .



6. Find the equation of the straight line passing through the intersection of the straight lines  $2x - 7y + 11 = 0$  and  $x + 3y - 8 = 0$  if it

(a) passes through the origin ;

(b) is perpendicular to the straight line  $2x - 5y + 6 = 0$  ;

(c) makes equal intercepts on the two axes.

7. Prove that the straight line  $3x + 4y + 7 = 0$  touches the circle  $x^2 + y^2 - 4x - 6y - 12 = 0$  and find the co-ordinates of the point of contact.

8. Find the focus, vertex and directrix of the parabola

$$(y + 3)^2 = 2(x + 2).$$

9. What do you understand by the term 'eccentricity' as applied to a hyperbola ?

Find the equation of the hyperbola whose focus is  $(2, 3)$ , and directrix, the line  $x + 2y = 1$  and eccentricity  $\sqrt{3}$ .

### GROUP C

10. Give instances from the sides and edges of a cube of :

(a) parallel planes, (b) planes perpendicular to one another, (c) lines parallel to a plane, (d) lines perpendicular to a plane, (e) pairs of skew lines.

Or, Prove that if a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

11. The volume of a right prism is 80 cu. ft. and its base is a triangle whose sides are 3 ft., 4 ft. and 5 ft. respectively. Find the height and the area of the total surface of the prism.

Or, A conical tent is required to accommodate 4 people ; each person must have 20 sq. ft. of space on the ground and 100 cu. ft. of air to breathe. Find the height and radius of the tent. [ $\pi = \frac{22}{7}$ .]













